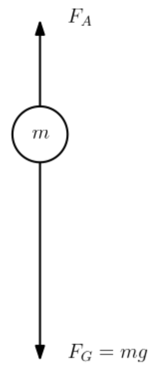
THE FALLING BODY



Here we will consider an object of mass m at some height above the ground. The object can be dropped or thrown straight up and then let fall to the ground subject to the force of gravity F_G . There will be air resistance F_A that is proportional to the magnitude of the velocity and will always act opposite to the velocity v.

Our direction convention will be positive v is up. We know from Newton's second law that $F = m\,a$ or $F = m \cdot \frac{\mathrm{d}v}{\mathrm{d}t}$. This will always be the LHS of our equation. The RHS will depend on whether the ball is dropped or given an initial upward velocity.

So, our equation for the falling ball will be

$$m \cdot \frac{\mathrm{d}v}{\mathrm{d}t} = \pm F_A - mg$$

with a minus sign in front of mg since positive motion is up. When the motion is up $v > 0$ we will have $-F_A$ and when the motion is downward $+F_A$ with $v < 0$.

The downward force due to gravity mg will always be negative in this example since we defined positive is up. What do we do with F_A ? We have two conditions to consider.

If v is positive (object going up), then the air resistance will be negative, pushing against the object and slowing it down.

$$m \cdot \frac{\mathrm{d}v}{\mathrm{d}t} = -F_A - mg$$

Next, when the object is falling downward then we will have $+F_A$, i.e. pushing up (positive direction) on the object. But, as we said above the air resistance is proportional to the velocity. Since going downward, v is negative so we will again have

$$m \cdot \frac{dv}{dt} = -F_A - mg$$

as our equation of motion. This is a special case and with other conditions you may need different differential equations for the upward and downward segments.

Example 1. Let m = 50 kg, air resistance = 5 v and $g = \frac{9.8\ m}{s^2}$. The object is thrown straight up from a platform 100m above the ground at $\frac{10\ m}{s}$. Find v when it hits the ground.

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restart
with(plots) :
with(DETools) :
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First our initial condition.

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ics1 := v(0) = 10;
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$$ics1 := v(0) = 10 \qquad \textbf{(1)}$$

Now define a function that is the RHS of our differential equation. The mass has been divided out.

$$\frac{dv}{dt} = -\frac{F_A}{m} - g$$

```
f := (t, y)→−0.1·y − 9.8;
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$$f := (t, y) \mapsto -0.1 \cdot y - 9.8 \qquad \textbf{(2)}$$

Next, substitute $v(t)$ into our dummy equation to get a differential equation for this situation. Then solve it with our initial condition.

```
de1 := diff(v(t), t) =−0.1 v(t) − 9.8;
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$$de1 := \frac{d}{dt} v(t) = -0.1\ v(t) - 9.8 \qquad \textbf{(3)}$$

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sol1 := dsolve({ics1, de1});
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$$sol1 := v(t) = -98 + 108\ e^{-\frac{t}{10}} \qquad \textbf{(4)}$$

Now, to calculate v when it hits the ground, we need to know when it hits the ground. We can get the when from how far it falls to hit the ground. We can integrate our *sol1* to get an expression for position or $s(t)$.

$s := int(rhs(sol1), t);$

$$s := -98\,t - 1080\,\mathrm{e}^{-\frac{t}{10}} \tag{5}$$

Maple does not add a constant of integration, so we need to add it to the expression and then find out what it is.

$s := s + c;$

$$s := -98\,t - 1080\,\mathrm{e}^{-\frac{t}{10}} + c \tag{6}$$

Since our point of reference is the platform at 100m and time of zero.

$c := eval(\ solve(s=0, c), t=0);$

$$c := 1080 \tag{7}$$

Check to make sure the expression is what we want. Then evaluate it for t when it falls 100m.

$s;$

$$-98\,t - 1080\,\mathrm{e}^{-\frac{t}{10}} + 1080 \tag{8}$$

$v\_ground := evalf(solve(s=-100, t))\ ;$

$$v\_ground := 5.981471333,\ -3.32203508 \tag{9}$$

We obtain two solutions but only the positive one makes sense. So now we can get the velocity when the body hits the ground.

$eval(sol1, t=v\_ground[1]);$

$$v(5.981471333) = -38.61841902 \tag{10}$$

This looks a little strange, but remember that positive is up and the body was falling down so v will be negative.

Example 2. Suppose an object with mass 10 kg is launched upward with initial velocity $\frac{20\ m}{s}$ from a platform that is 3m high. Suppose there is a force due to air resistance of magnitude v directed opposite to the velocity, where the velocity v is measured in m/s. Find the maximum height above the ground that the object reaches. Our position reference point is the ground (not the platform) this time.

Our initial condition is

$ics2 := v2(0) = 20;$

$$ics2 := v2(0) = 20 \tag{11}$$

Again, we divide out the mass and note we can reuse equation 2 from above.

$f2 := (t, y) \to -0.1 \cdot y - 9.8;$

$$f2 := (t, y) \mapsto -0.1 \cdot y - 9.8 \tag{12}$$

$de2 := diff(v2(t), t) = -0.1\ v2(t) - 9.8;$

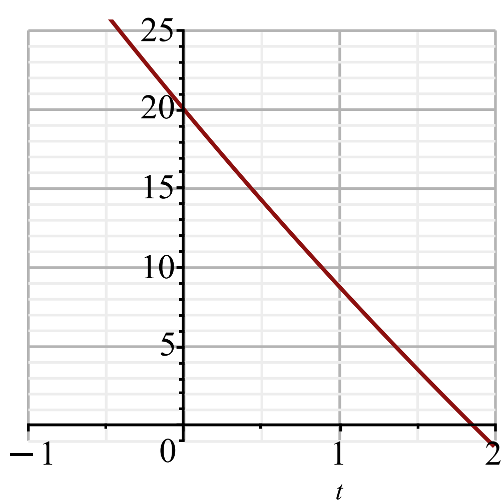
$$de2 := \frac{\mathrm{d}}{\mathrm{d}t}\ v2(t) = -0.1\ v2(t) - 9.8 \tag{13}$$

$sol4 := dsolve(\{ics2, de2\});$

$$sol4 := v2(t) = -98 + 118\ \mathrm{e}^{-\frac{t}{10}} \tag{14}$$

When the object reaches its maximum height, its velocity v will be zero. Equation (15) is the ball's velocity so let's plot it and see where it is zero. It is zero just under 2 and note at $t = 0$ we see the initial conditions of 20.

$plot(rhs(sol4), size = [200, 200], view = [-1..2, -1..25], gridlines);$



Now let's calculate the exact time the object's velocity is zero.

$t\_maxheight := evalf(solve(rhs(sol4) = 0, t));$

$$t\_maxheight := 1.857171458 \qquad \textbf{(15)}$$

Now we need position.

$s2 := int(rhs(sol4), t);$

$$s2 := -98\,t - 1180\,e^{-\frac{t}{10}} \qquad \textbf{(16)}$$

$s2 := s2 + d;$

$$s2 := -98\,t - 1180\,e^{-\frac{t}{10}} + d \qquad \textbf{(17)}$$

Our point of reference for time at zero is the platform at 3m.
$d := eval(\ solve(s2 = 3, d), t = 0);$

$$d := 1183 \qquad \textbf{(18)}$$

Check to make sure the expression is what we want.
$s2$

$$-98\,t - 1180\,e^{-\frac{t}{10}} + 1183 \qquad \textbf{(19)}$$

The expression *sol6* looks correct so plug in the time at maximum height *t_maxheight* so we can calculate the *max_height*.
$max\_height := eval(s2, t = t\_maxheight);$

$$max\_height := 20.997197 \qquad \textbf{(20)}$$