THE X-Y PROJECTILE MOTION WITH AIR RESISTANCE

Newton's second law of motion was used in The Falling Body example. An object with mass *m* was subject to the force of gravity and a force in the opposite direction that was proportional to the velocity of the body. It was an interesting and useful exercise, but only in one direction.

Now let's look at the two dimensional projectile problem with air resistance proportional to the mass *m* and the velocity of the object.

restart;

The y component looks familiar. We could call this component just y, but we want to reserve that for the direction.

$$y\_dir := m\cdot\left(\frac{d^2}{dt^2}y(t)\right) = -k\cdot m\cdot\left(\frac{d}{dt}y(t)\right) - m\cdot g;$$

$$y\_dir := m\left(\frac{d^2}{dt^2}\, y(t)\right) = -k\, m\left(\frac{d}{dt}\, y(t)\right) - m\, g \tag{1}$$

The x component is similar without the effect of gravity and we assume so air resistance in the x direction.

$$x\_dir := m\cdot\left(\frac{d^2}{dt^2}x(t)\right) = -k\cdot m\cdot\left(\frac{d}{dt}x(t)\right);$$

$$x\_dir := m\left(\frac{d^2}{dt^2}\, x(t)\right) = -k\, m\left(\frac{d}{dt}\, x(t)\right) \tag{2}$$

We can use *dsolve* to solve these equations. We haven't defined any initial conditions so Maple inserts its own arbitrary constants.

$dsolve(x\_dir); dsolve(y\_dir);$

$$x(t) = c_1 + c_2\, e^{-k\, t}$$

$$y(t) = -\frac{e^{-k\, t}\, c_1}{k} - \frac{g\, t}{k} + c_2 \tag{3}$$

For this exercise, let's have the projectile's lauch point as the origin. We will give it an initial velocity *v* at an angle Θ . The projectile's velocity will have a transverse *T* and upward *U* components. This will be written as

$$(T, U) = \left(v0\cdot\cos(\Theta), v0\cdot\sin(\Theta)\right).$$

Now we need to use these initial conditions when we solve the equations.

$dsolve(\ (\{x\_dir, x(0) = 0, \mathrm{D}(x)(0) = T\})\ );$
$dsolve(\ (\{y\_dir, y(0) = 0, \mathrm{D}(y)(0) = U\})\ );$

$$x(t) = -\frac{T\left(\mathrm{e}^{-k\,t} - 1\right)}{k}$$

$$y(t) = \frac{(-U\,k - g)\ \mathrm{e}^{-k\,t} + (-g\,t + U)\ k + g}{k^2} \qquad \textbf{(4)}$$

These are useful equations, but for future simplicity we really only need the right hand sides. So let's redo the solutions.

$X := rhs(dsolve(\ \{x\_dir, x(0) = 0, \mathrm{D}(x)(0) = T\}\ )\ );$

$$X := -\frac{T\left(\mathrm{e}^{-k\,t} - 1\right)}{k} \qquad \textbf{(5)}$$

$Y := rhs(dsolve(\ \{y\_dir, y(0) = 0, \mathrm{D}(y)(0) = U\}\ )\ );$

$$Y := \frac{(-U\,k - g)\ \mathrm{e}^{-k\,t} + (-g\,t + U)\ k + g}{k^2} \qquad \textbf{(6)}$$

Use 9.7 m^/s for g and theta in degrees. Here we will use the : operator. The values are easy to verify in the Variables dropdown in the left panel.

$g := 9.8 : v0 := 600 : \mathrm{theta} := 60 :$

$T := v0\cdot\cos(convert(\mathrm{theta}\cdot degrees, radians)\ );$

$$T := 300 \qquad \textbf{(7)}$$

$U := v0\cdot\sin(convert(\mathrm{theta}\cdot degrees, radians)\ );$

$$U := 300\sqrt{3} \qquad \textbf{(8)}$$

Now we can look at our solutions that are dependent on k and t.

$X; Y;$

$$-\frac{300\left(e^{-k t}-1\right)}{k}$$

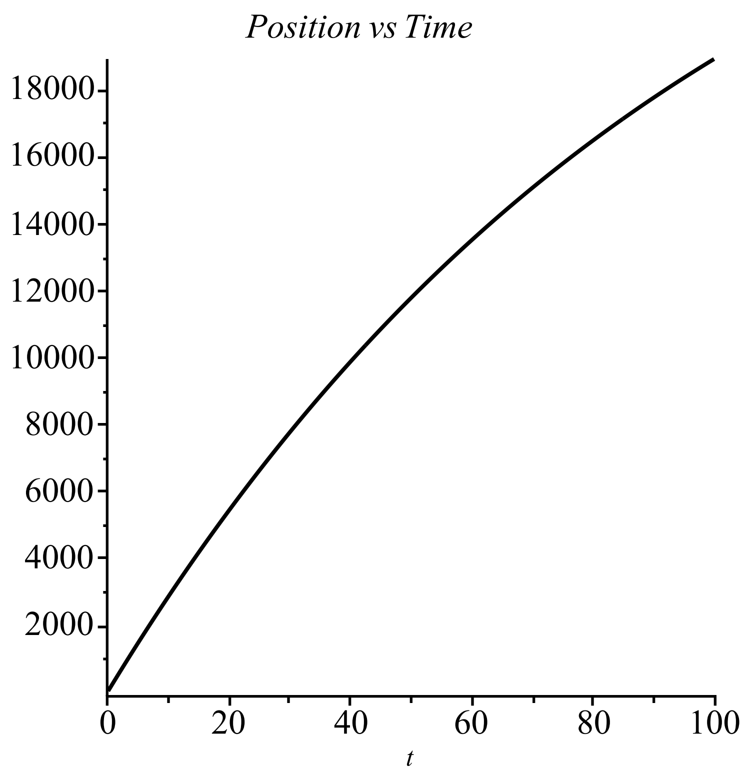
$$\frac{\left(-300\sqrt{3}\,k-9.8\right)e^{-k t}+\left(-9.8\,t+300\sqrt{3}\right)k+9.8}{k^2} \qquad \textbf{(9)}$$

So what do these solutions look like. We will let $k = k0$ and set it equal to 0.01.

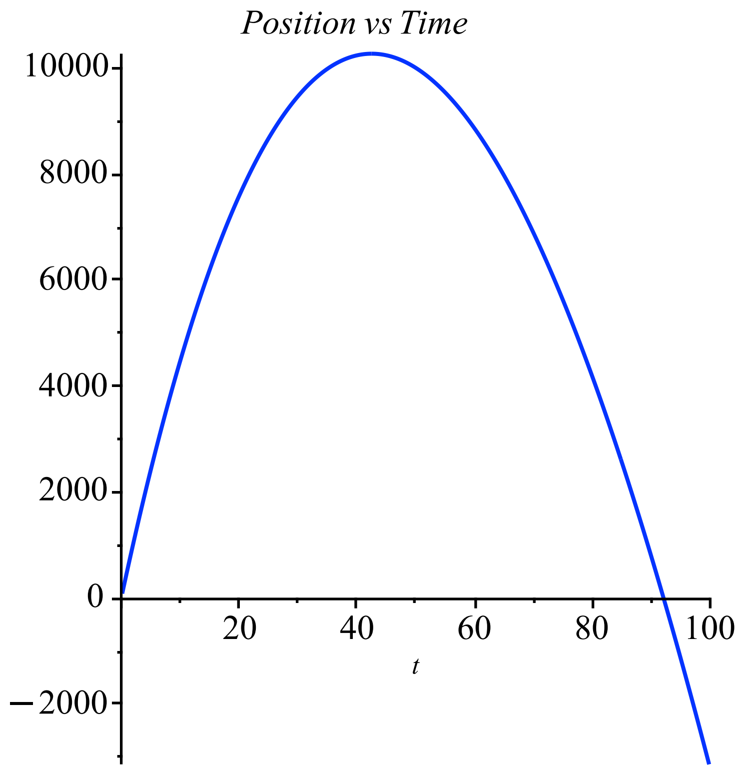
First plot the transverse or X component.

$k0 := 0.01 :$

$plot(\{subs(k=k0, X)\}, t=0..100, title='Position\ vs\ Time', color=black, size=[300, 300]);$

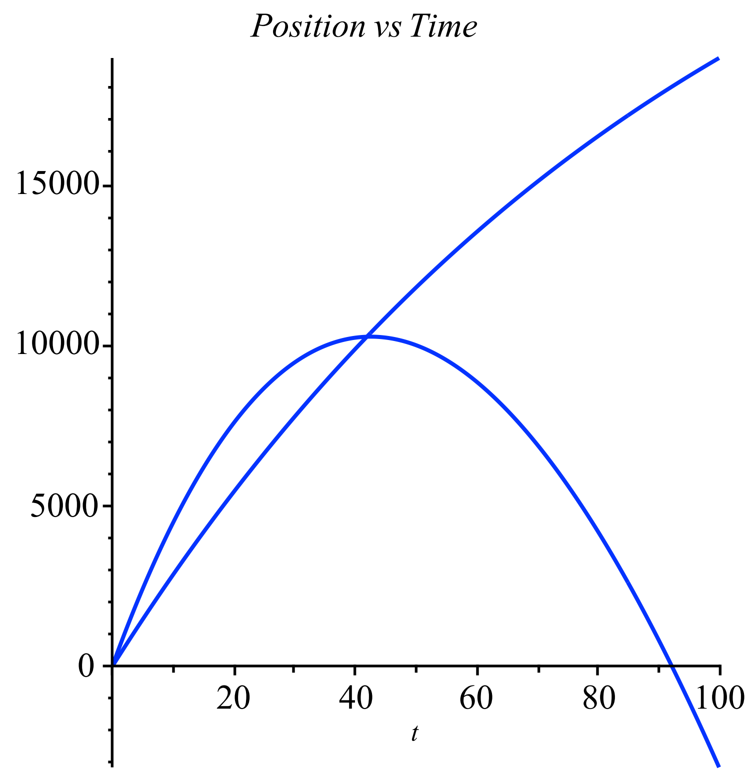


$plot(\{subs(k=k0, Y)\}, t=0..100, title='Position\ vs\ Time', color=blue, size=[300, 300]);$

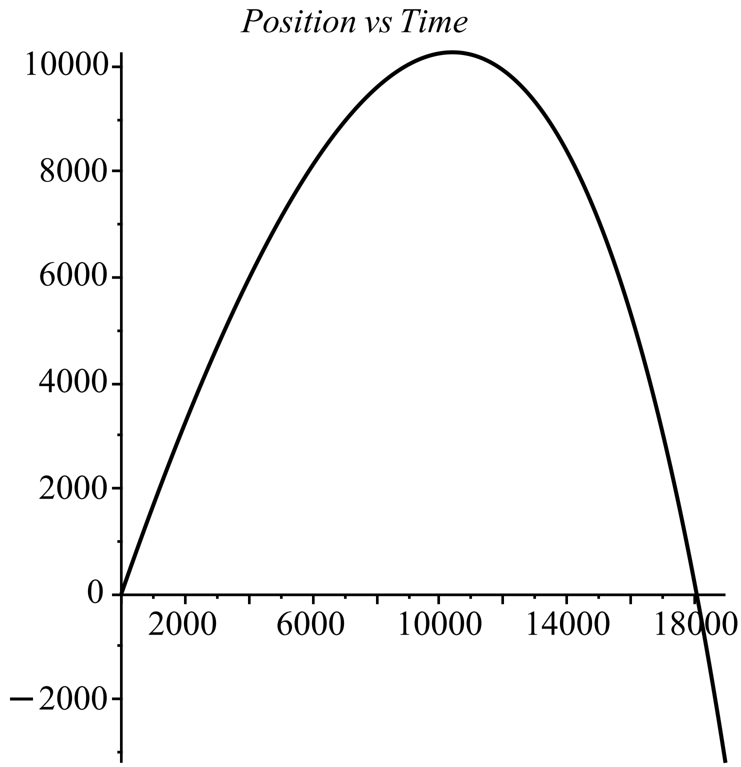


These are interesting, but plotting the components separately we don't see the effect of air resistance. Let's try plotting this parametrically Y versus X. Note the placement of brackets here. First we will place the brackets around the two *subs* commands.

$plot([subs(k = k0, X), subs(k = k0, Y)], t = 0..100, title = 'Position\ vs\ Time', color = blue, size = [300, 300]);$



plot([*subs*(*k* = *k0*, *X*), *subs*(*k* = *k0*, *Y*) , *t* = 0 ..100], *title* ='*Position vs Time*', *color* = *black*, *size* = [300, 300]);



Notice the air resistance effect. This plot is not a parabola now.

From the above Y plot that the altitude returns to zero between 80 and 100 seconds after launch. So let's use *fsolve* to get an exact number for our time of flight.

$T := fsolve(subs(k = k0, Y) = 0, t = 80..100);$

$$T := 92.10191668 \quad \textbf{(10)}$$

Now plug this T into X to calculate the range when it hits the ground.

$subs(\{k = k0, t = T\}, X);$

$$-9210.191668\, e^{-0.9210191668} + 9210.191668 \quad \textbf{(11)}$$

Oops, we don't need to get out a calculator. We can get Maple to do it for us with the *evalf* function.

$evalf(subs(\{k = k0, t = T\}, X));$

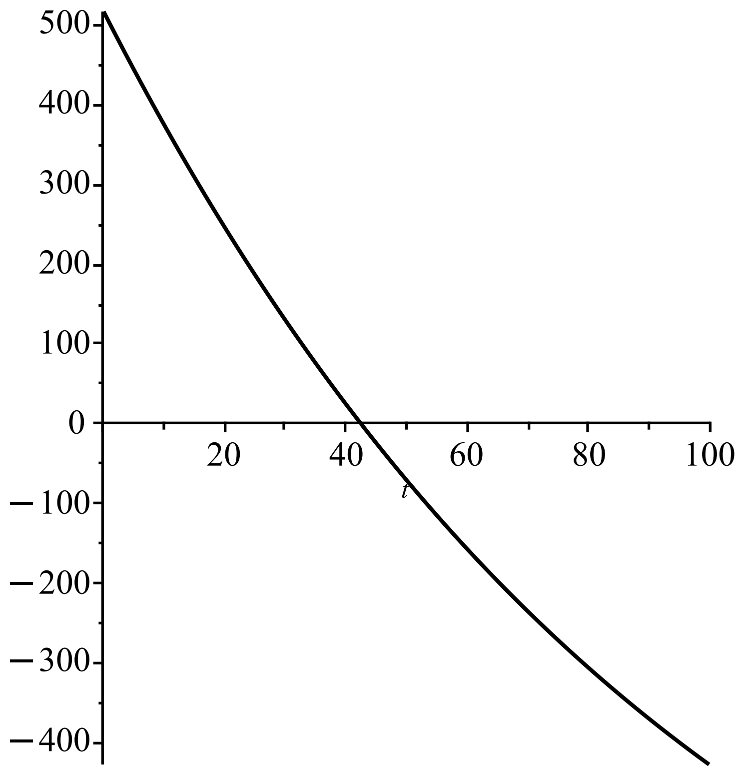
$$5543.493798 \quad \textbf{(12)}$$

Let's find the rise time. Thinking about this problem, at the top of the path, the vertical velocity is zero. So first define the vertical velocity.

$v := \mathit{diff}(Y, t);$

$$v := \frac{-\left(-300\sqrt{3}\,k - 9.8\right) k\,\mathrm{e}^{-k\,t} - 9.8\,k}{k^2} \qquad \textbf{(13)}$$

$\mathit{plot}(\mathit{subs}(k = k0, v), t = 0..100, \mathit{color} = \mathit{black}, \mathit{size} = [300, 300]);$



The rise time will be where this curve hits zero so let *fsolve* find that zero.

$\mathit{Trise} := \mathit{fsolve}(\mathit{subs}(k = k0, v));$

$$\mathit{Trise} := 42.54112774 \qquad \textbf{(14)}$$

The fall time is then total time T - Trise.

$\mathit{Tfall} := T - \mathit{Trise};$

$$\mathit{Tfall} := 49.56078894 \qquad \textbf{(15)}$$

Note if we plug our rise time *Trise* into our equation for *Y*, we will obtain the max altitude achieved.

$\mathit{evalf}(\mathit{subs}(\{k = k0, t = \mathit{Trise}\}, Y));$

$$10271.21907 \qquad \textbf{(16)}$$

From the original reference, now plot multiples curves on one plot.

```
plot( {seq( [subs(k = n·0.01, X), subs(k = n·0.01, Y), t = 0 ..100], n = 1 ..5) }, title ='Projectile Motion
    ', labels = ['X','Y']);
```