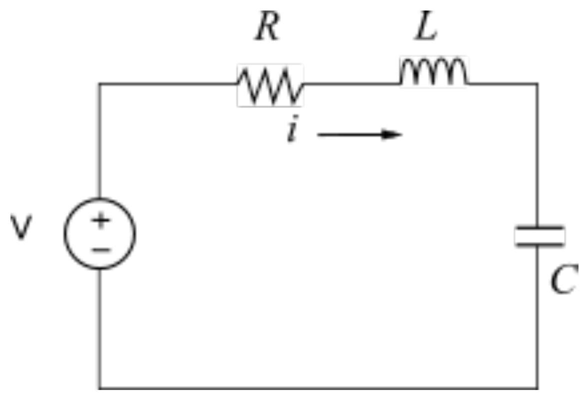
THE RLC PROBLEM



$VR := R \cdot i(t);$

$$VR := R\, i(t) \quad (1)$$

$VL := L \cdot \frac{d}{d\,t} i(t);$

$$VL := L \left(\frac{d}{dt}\, i(t) \right) \quad (2)$$

$VC := \frac{\int_0^t i(s)\, ds}{C};$

$$VC := \frac{\int_0^t i(s)\, ds}{C} \quad (3)$$

Kirchhoff's voltage law says that the sum of the voltages around a closed loop is zero.

$KVL := VR + VL + VC = V;$

$$KVL := R\, i(t) + L \left(\frac{d}{dt}\, i(t) \right) + \frac{\int_0^t i(s)\, ds}{C} = V \quad (4)$$

Now need to differentiate to clear the integral in the equation.

$KVL := diff(KVL, t);$

$$KVL := R\left(\frac{d}{dt}\, i(t)\right) + L\left(\frac{d^2}{dt^2}\, i(t)\right) + \frac{i(t)}{C} = 0 \qquad \textbf{(5)}$$

We have a second order differential equation and need two initial conditions. The first is straightforward, let $i(0) = 0$ since the inductor resists the flow of current.

Now, the capacitor is a short at $t = 0$ so we have an RL circuit at $t = 0$. We can use that observation to determine a second initial condition.

$IC\_RL := i(0) = 0;$

$$IC\_RL := i(0) = 0 \qquad \textbf{(6)}$$

$K\_RL := VR + VL = VRL;$

$$K\_RL := R\, i(t) + L\left(\frac{d}{dt}\, i(t)\right) = VRL \qquad \textbf{(7)}$$

$sol\_RL := dsolve(\{K\_RL, IC\_RL\}, i(t));$

$$sol\_RL := i(t) = -\frac{VRL\left(e^{-\frac{R\,t}{L}} - 1\right)}{R} \qquad \textbf{(8)}$$

So the current is the RHS of this solution.

$i\_RL := rhs(sol\_RL);$

$$i\_RL := -\frac{VRL\left(e^{-\frac{R\,t}{L}} - 1\right)}{R} \qquad \textbf{(9)}$$

Now we can differentiate this current at $t = 0$ for the second initial condition.

$i0prime := simplify(subs(t = 0, diff(i\_RL, t)));$

$$i0prime := \frac{VRL}{L} \qquad \textbf{(10)}$$

Note that simplify is needed, otherwise you would be left with an e^0 term in the solution. Nothing magic here since $V = L \cdot \frac{d}{dt} i(t)$ so $\frac{d}{dt} i(t) = \frac{V}{L}$.

Now we are ready to solve the RLC circuit.

$IC\_RLC := i(0) = 0, D(i)(0) = i0prime;$

$$IC\_RLC := i(0) = 0, D(i)(0) = \frac{VRL}{L} \qquad \textbf{(11)}$$

$sol\_RLC := dsolve(\{KVL, IC\_RLC\}, i(t));$

$$sol\_RLC := i(t) = \frac{\sqrt{C\left(R^2 C - 4 L\right)}\ VRL \left(e^{-\frac{\left(R C - \sqrt{C\left(R^2 C - 4 L\right)}\right) t}{2 C L}} - e^{-\frac{\left(R C + \sqrt{C\left(R^2 C - 4 L\right)}\right) t}{2 C L}}\right)}{R^2 C - 4 L} \qquad \textbf{(12)}$$

There are some interesting observations about what is going on with this solution. The $\frac{R}{L}$ components in the exponentials controls the damping. The $\sqrt{C\left(R^2 C - 4 L\right)}$ term rewritten as $\sqrt{R^2 C^2 - 4 LC}$ controls whether the system is under, critically, or over - damped. When this square root has a negative sign under it, the roots are complex and the system will exhibit sinusoidal behavior.

So, if $L = 10^{-6}$ and $C = 10^{-9}$ then for critical damping $R =$

$$evalf\left(\sqrt{\frac{4 \cdot 10^{-6}}{10^{-9}}}\right)$$

$$63.24555320 \qquad \textbf{(13)}$$

$with(plots):$

$i\_1 := simplify\left(subs\left(VRL = 1, C = 10^{-9}, R = 63.246, L = 10^{-6}, rhs(sol\_RLC)\right)\right);$

$$i\_1 := 4.206436061\, e^{-3.150413453 \times 10^{7}\, t} - 4.206436061\, e^{-3.174186547 \times 10^{7}\, t} \qquad \textbf{(14)}$$

$plt1 := plot\left(100 \cdot i\_1, t = 0 ..2 \cdot 10^{-6}, gridlines, view = \left[0 ..8. \cdot 10^{-7}, -0.1 ..0.1\right]\right):$

$i\_2 := simplify\left(subs\left(VRL = 1, C = 10^{-9}, R = 50, L = 10^{-6}, rhs\left(sol\_RLC\right)\right)\right);$

$$i\_2 := -\frac{\mathrm{I}}{150}\left(\mathrm{e}^{5000000\,(\mathrm{I}\sqrt{15}-5)\,t} - \mathrm{e}^{-5000000\,(\mathrm{I}\sqrt{15}+5)\,t}\right)\sqrt{15} \qquad \textbf{(15)}$$

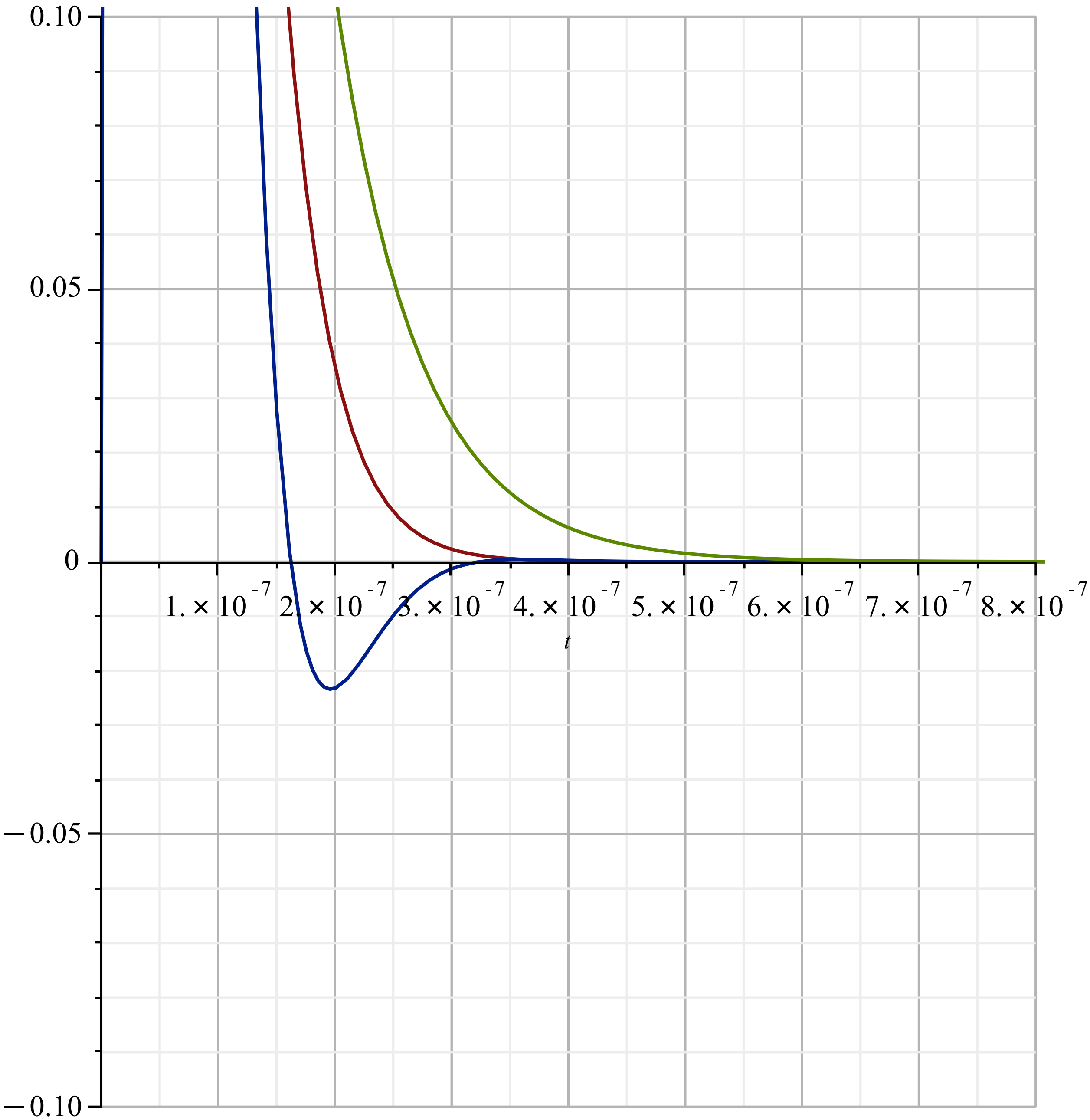
$plt2 := plot\left(100\cdot i\_2, t = 0\,..2\cdot 10^{-6}, gridlines, view = \left[0\,..8.\cdot 10^{-7}, -0.1\,..0.1\right]\right):$

$i\_3 := simplify\left(subs\left(VRL = 1, C = 10^{-9}, R = 85, L = 10^{-6}, rhs\left(sol\_RLC\right)\right)\right);$

$$i\_3 := \frac{\sqrt{129}\left(\mathrm{e}^{2500000\,(-17+\sqrt{129})\,t} - \mathrm{e}^{-2500000\,(17+\sqrt{129})\,t}\right)}{645} \qquad \textbf{(16)}$$

$plt3 := plot\left(100\cdot i\_3, t = 0\,..2\cdot 10^{-6}, gridlines, view = \left[0\,..8.\cdot 10^{-7}, -0.1\,..0.1\right]\right):$

$display\left(plt1, plt2, plt3\right);$



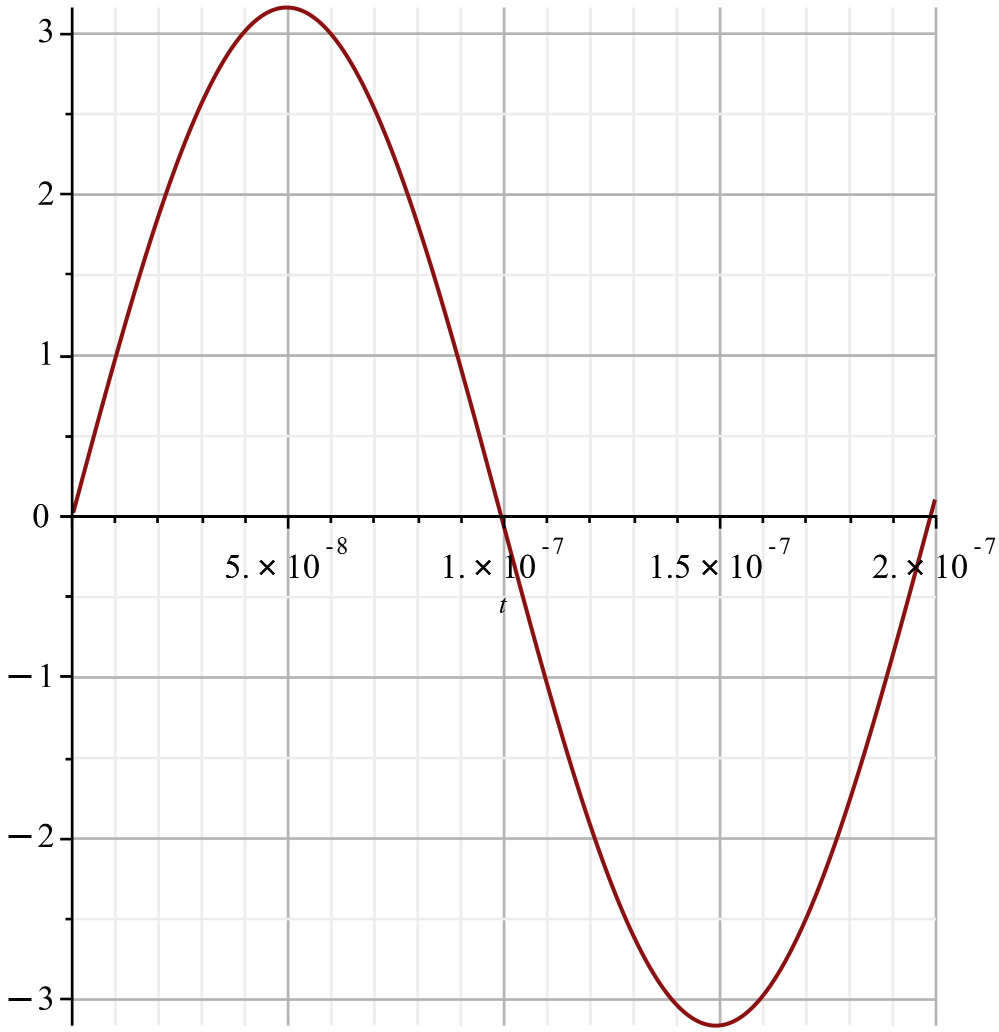
In the plot above, the middle line is the critically damped. The green one to the right is overdamped. The blue one to the left is underdamped and starting to show sinusoidal behavior due to the complex roots.

Let's set $R = 0$ and see what happens.

$i\_4 := simplify\left(subs\left(VRL=1, C=10^{-9}, R=0, L=10^{-6}, rhs(sol\_RLC)\right)\right);$

$$i\_4 := \frac{\sqrt{10}\,\sin\left(10000000\sqrt{10}\,t\right)}{100} \qquad \textbf{(17)}$$

$plot\left(100 \cdot i\_4, t=0..0.2 \cdot 10^{-6}, gridlines, size=[400, 400]\right);$



From the graph the period looks like $2 \cdot 10^{-7}$ or 5 megahertz. Remember $\omega = 2 \cdot \pi \cdot f$ or $31.4 \cdot 10^{6}$ radians/sec. The actual value is found from equation (12) above with setting $R=0$. After simplification you get terms with $e^{\frac{\left(\sqrt{-4\,LC}\right)}{LC}}$ which simplifies to $e^{\frac{i}{\sqrt{LC}}}$ so the resonant frequency is equal to $\frac{1}{\sqrt{LC}}$.

So for our example the frequency is $31.6 \cdot 10^{6}$ radians/sec.