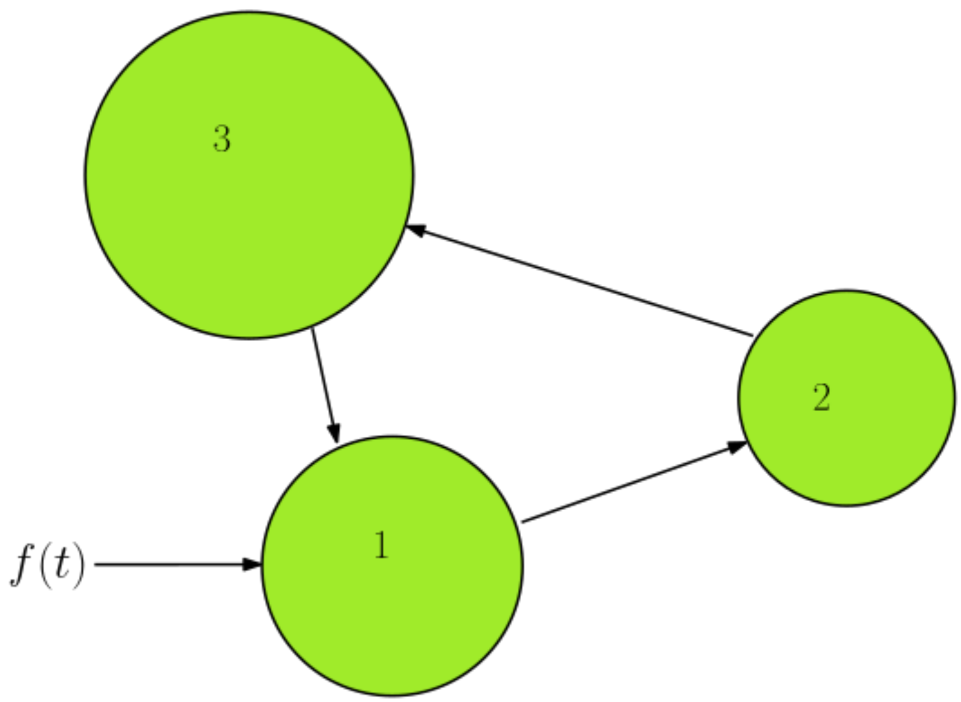
THE MULTIPLE MIXING TANK PROBLEM

I found the following example at https://www.math.utah.edu/~gustafso/2250systems-de.pdf.

This link is Chapter 11 Systems of Differential Equations. I tried unsuccessfully to find the rest of the book so I could credit it. It is a really interesting example and Maple makes it very to solve and visualize.

To look at multiple mixing tanks let's continue the pond theme. Here, we have three ponds connected by streams. Pond 1 has a pollutions source. The goal is the determine the amount of pollution in each pond.



We will assume the following.

- The pollutant flows into pond 1 at 0.125 lb/min for 2880 minutes, then zero.
- The flow rates out of each pond are f_1, f_2, f_3 in gal/min. Each pond is well mixed.
- The three ponds have volumes V_1, V_2, V_3 in gal which remain constant.
- The amount of pollution in each pond is $p_1(t), p_2(t), p_3(t)$ in lbs.

The pollutant flux out of each pond will be the flow rate in $\frac{gal}{\min}$ times the pollutant concentration in $\frac{pounds}{Volume}$. For pond 1 this would be $f_1(t) \cdot \left(\frac{p_1}{V_1}\right)$.

We can now write down the differential equations for this example.

restart

$pf_1 := 100;\ pf_2 := 100;\ pf_3 := 100;$

$$pf_1 := 100$$

$$pf_2 := 100$$

$$pf_3 := 100 \tag{1}$$

$V_1 := 100000;\ V_2 := 100000;\ V_3 := 100000;$

$$V_1 := 100000$$

$$V_2 := 100000$$

$$V_3 := 100000 \tag{2}$$

$f(t) := 0.125;$

$$f := t \mapsto 0.125 \tag{3}$$

$Pond1 := \frac{d}{dt} p_1(t) = \left(\frac{pf_3}{V_3}\right) \cdot p_3(t) - \left(\frac{pf_1}{V_1}\right) \cdot p_1(t) + f(t);$

$Pond2 := \frac{d}{dt} p_2(t) = \left(\frac{pf_1}{V_1}\right) \cdot p_1(t) - \left(\frac{pf_2}{V_2}\right) \cdot p_2(t);$

$Pond3 := \frac{d}{dt} p_3(t) = \left(\frac{pf_2}{V_2}\right) \cdot p_2(t) - \left(\frac{pf_3}{V_3}\right) \cdot p_3(t);$

$$Pond1 := \frac{d}{dt} p_1(t) = \frac{p_3(t)}{1000} - \frac{p_1(t)}{1000} + 0.125$$

$$Pond2 := \frac{d}{dt} p_2(t) = \frac{p_1(t)}{1000} - \frac{p_2(t)}{1000}$$

$$Pond3 := \frac{d}{dt}\, p_3(t) = \frac{p_2(t)}{1000} - \frac{p_3(t)}{1000} \tag{4}$$

For a numerical example we will let each pond's $\frac{f_i}{V_i}$ term equal 0.001. As you will see below, the pond pollution trajectories are interesting in the first 2880 minutes and then converge to the same value in each pond.

$PondSystem := \{Pond1, Pond2, Pond3, p_1(0) = 0,\ p_2(0) = 0,\ p_3(0) = 0\};$

$$PondSystem := \left\{ \frac{d}{dt}\, p_1(t) = \frac{p_3(t)}{1000} - \frac{p_1(t)}{1000} + 0.125, \frac{d}{dt}\, p_2(t) = \frac{p_1(t)}{1000} - \frac{p_2(t)}{1000}, \frac{d}{dt}\, p_3(t) = \frac{p_2(t)}{1000} - \frac{p_3(t)}{1000}, p_1(0) = 0, p_2(0) = 0, p_3(0) = 0 \right\} \tag{5}$$

$PondSolution := dsolve(PondSystem);$

$$PondSolution := \left\{ p_1(t) = \frac{t}{24} + \frac{125}{3} + \frac{125\sqrt{3}\, e^{-\frac{3t}{2000}} \sin\left(\frac{\sqrt{3}\, t}{2000}\right)}{9} - \frac{125\, e^{-\frac{3t}{2000}} \cos\left(\frac{\sqrt{3}\, t}{2000}\right)}{3}, p_2(t) = -\frac{250\sqrt{3}\, e^{-\frac{3t}{2000}} \sin\left(\frac{\sqrt{3}\, t}{2000}\right)}{9} + \frac{t}{24}, p_3(t) = -\frac{125}{3} + \frac{125\sqrt{3}\, e^{-\frac{3t}{2000}} \sin\left(\frac{\sqrt{3}\, t}{2000}\right)}{9} + \frac{125\, e^{-\frac{3t}{2000}} \cos\left(\frac{\sqrt{3}\, t}{2000}\right)}{3} + \frac{t}{24} \right\} \tag{6}$$

$P1 := eval(p_1(t), PondSolution);$

$$P1 := \frac{t}{24} + \frac{125}{3} + \frac{125\sqrt{3}\, e^{-\frac{3t}{2000}} \sin\left(\frac{\sqrt{3}\, t}{2000}\right)}{9} - \frac{125\, e^{-\frac{3t}{2000}} \cos\left(\frac{\sqrt{3}\, t}{2000}\right)}{3} \tag{7}$$

$P2 := eval(p_2(t), PondSolution);$

$$P2 := -\frac{250\sqrt{3}\,e^{-\frac{3t}{2000}}\sin\left(\frac{\sqrt{3}\,t}{2000}\right)}{9}+\frac{t}{24} \tag{8}$$

$P3 := eval\left(p_3(t), PondSolution\right);$

$$P3 := -\frac{125}{3}+\frac{125\sqrt{3}\,e^{-\frac{3t}{2000}}\sin\left(\frac{\sqrt{3}\,t}{2000}\right)}{9}+\frac{125\,e^{-\frac{3t}{2000}}\cos\left(\frac{\sqrt{3}\,t}{2000}\right)}{3}+\frac{t}{24} \tag{9}$$

evalf(eval(P1, t = 2880)); evalf(eval(P2, t = 2880)); evalf(eval(P3, t = 2880));

$$162.3016570$$

$$119.6140523$$

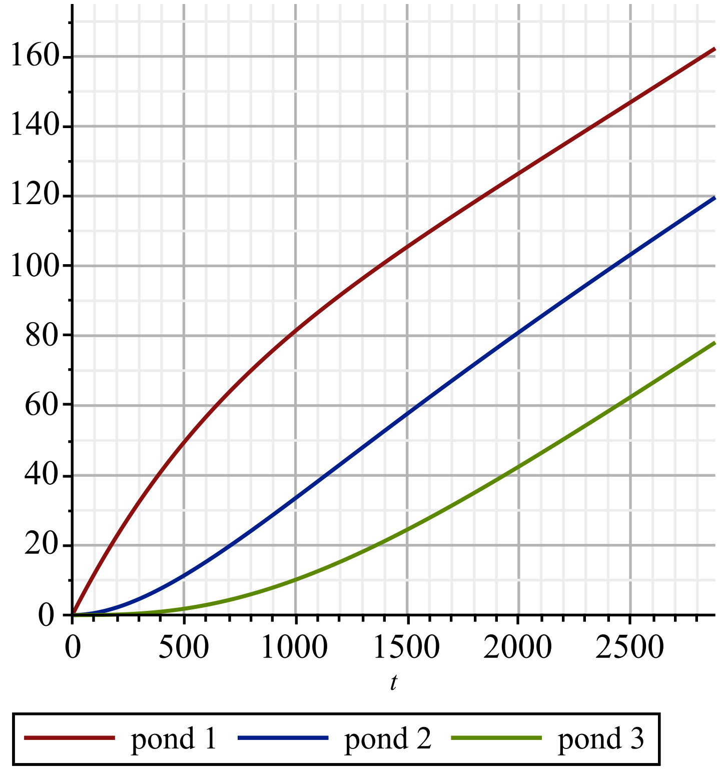
$$78.08429066 \tag{10}$$

Let's see what the pollution in each pond looks like in the first 2880 minutes.

with(*DETools*) :
sol1 := *dsolve*(*eval*(*PondSystem*), *numeric*);

$$sol1 := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \tag{11}$$

$plots\text{:-}odeplot\left(sol1, \left[\left[t, p_1(t)\right], \left[t, p_2(t)\right], \left[t, p_3(t)\right]\right], t=0..2880, legend=\left[\text{"pond 1"}, \text{"pond 2"}, \text{"pond 3"}\right], gridlines, view=\left[0..2880, 0..175\right], size=\left[300, 300\right]\right);$



Now, turn off the pollution going into pond 1 and resolve the equations for the Long Term (LT) solution.

$Pond1 := lhs(Pond1) = rhs(Pond1) - 0.125;$

$$Pond1 := \frac{d}{dt}\, p_1(t) = \frac{p_3(t)}{1000} - \frac{p_1(t)}{1000} \tag{12}$$

$Pond1$

$$\frac{d}{dt}\, p_1(t) = \frac{p_3(t)}{1000} - \frac{p_1(t)}{1000} \tag{13}$$

$PondSystemLT := \left\{Pond1, Pond2, Pond3, p_1(0) = 162.302,\ p_2(0) = 119.614,\ p_3(0) = 78.085\right\};$

$$PondSystemLT := \left\{\frac{d}{dt}\, p_1(t) = \frac{p_3(t)}{1000} - \frac{p_1(t)}{1000}, \frac{d}{dt}\, p_2(t) = \frac{p_1(t)}{1000} - \frac{p_2(t)}{1000}, \frac{d}{dt}\, p_3(t) = \frac{p_2(t)}{1000} - \frac{p_3(t)}{1000}, p_1(0) = 162.302, p_2(0) = 119.614, p_3(0) = 78.085\right\} \tag{14}$$

$PondSolutionLT := dsolve(PondSystemLT);$

$$PondSolutionLT := \left\{p_1(t) = \frac{360001}{3000} - \frac{13843\sqrt{3}\,e^{-\frac{3t}{2000}}\sin\left(\frac{\sqrt{3}\,t}{2000}\right)}{1000} + \frac{25381\,e^{-\frac{3t}{2000}}\cos\left(\frac{\sqrt{3}\,t}{2000}\right)}{600}, p_2(t) = \frac{84217\sqrt{3}\,e^{-\frac{3t}{2000}}\sin\left(\frac{\sqrt{3}\,t}{2000}\right)}{3000} - \frac{1159\,e^{-\frac{3t}{2000}}\cos\left(\frac{\sqrt{3}\,t}{2000}\right)}{3000} + \frac{360001}{3000}, p_3(t) = -\frac{5336\sqrt{3}\,e^{-\frac{3t}{2000}}\sin\left(\frac{\sqrt{3}\,t}{2000}\right)}{375} - \frac{62873\,e^{-\frac{3t}{2000}}\cos\left(\frac{\sqrt{3}\,t}{2000}\right)}{1500} + \frac{360001}{3000}\right\} \tag{15}$$

$P1LT := eval(p_1(t), PondSolutionLT);$

$$P1LT := \frac{360001}{3000} - \frac{13843\sqrt{3}\,e^{-\frac{3t}{2000}}\sin\left(\frac{\sqrt{3}\,t}{2000}\right)}{1000} + \frac{25381\,e^{-\frac{3t}{2000}}\cos\left(\frac{\sqrt{3}\,t}{2000}\right)}{600} \tag{16}$$

$P2LT := eval(p_2(t), PondSolutionLT);$

$$P2LT := \frac{84217\sqrt{3}\,e^{-\frac{3t}{2000}}\sin\left(\frac{\sqrt{3}\,t}{2000}\right)}{3000} - \frac{1159\,e^{-\frac{3t}{2000}}\cos\left(\frac{\sqrt{3}\,t}{2000}\right)}{3000} + \frac{360001}{3000} \tag{17}$$

$P3LT := eval(p_3(t), PondSolutionLT);$

$$P3LT := -\frac{5336\sqrt{3}\,e^{-\frac{3t}{2000}}\sin\left(\frac{\sqrt{3}\,t}{2000}\right)}{375} - \frac{62873\,e^{-\frac{3t}{2000}}\cos\left(\frac{\sqrt{3}\,t}{2000}\right)}{1500} + \frac{360001}{3000} \tag{18}$$

Now you can see what the LT solutions are.

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evalf(eval(P1LT, t = 4000)); evalf(eval(P2LT, t = 4000)); evalf(eval(P3LT, t = 4000));
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$$119.9197210$$

$$119.9630419$$

$$120.1182371 \tag{19}$$

$sol2 := dsolve(eval(PondSystemLT), numeric);$

$$sol2 := \mathbf{proc}(x\_rkf45) \ \ldots \ \mathbf{end\ proc} \tag{20}$$

$plots\text{:-}odeplot(sol2, [[t, p_1(t)], [t, p_2(t)], [t, p_3(t)]], t = 0..5000, legend = [\text{"pond 1"}, \text{"pond 2"}, \text{"pond 3"}], gridlines, view = [0..5000, 0..175], size = [300, 300]);$