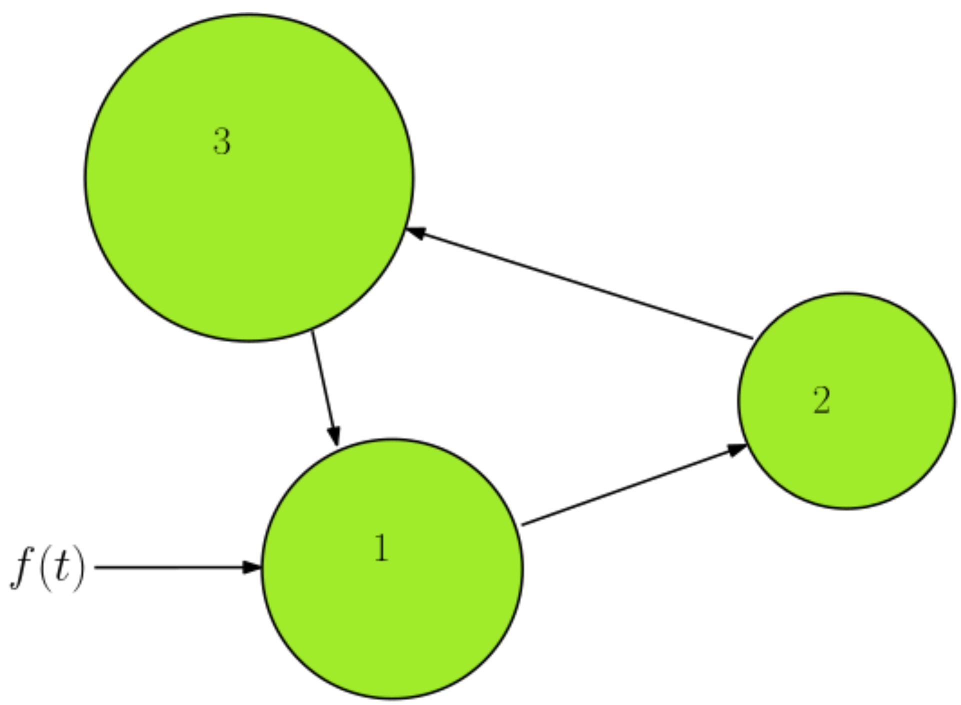
THE MULTIPLE MIXING TANK PROBLEM REVISITED

Now let's revisit the pond and pollution problem and remove the constraint that the flux $\frac{flow}{volume}$ is equal for each pond. What is really happening in the system long term?

I found the following example at https://www.math.utah.edu/~gustafso/2250systems-de.pdf.

This link is Chapter 11 Systems of Differential Equations. I tried unsuccessfully to find the rest of the book so I could credit it. It is a really interesting example and Maple makes it very to solve and visualize.

To look at multiple mixing tanks let's continue the pond theme. Here, we have three ponds connected by streams. Pond 1 has a pollutions source. The goal is the determine the amount of pollution in each pond.



We will assume the following.

• The pollutant flows into pond 1 at 1 lb/min.

• The flow rates out of each pond are f_1, f_2, f_3 in gal/min. Each pond is well mixed.

• The three ponds have volumes V_1, V_2, V_3 in gal which remain constant.

• The amount of pollution in each pond is $p_1(t), p_2(t), p_3(t)$ in lbs.

The pollutant flux out of each pond will be the flow rate in $\frac{gal}{\min}$ times the pollutant concentration in $\frac{pounds}{Volume}$. For pond 1 this would be $f_1(t) \cdot \left(\frac{p_1}{V_1}\right)$.

We can now write down the differential equations for this example.

restart

$pf_1 := 20;\ pf_2 := 10;\ pf_3 := 10;$

$$pf_1 := 20$$
$$pf_2 := 10$$
$$pf_3 := 10 \qquad \textbf{(1)}$$

$V_1 := 2000;\ V_2 := 5000;\ V_3 := 100;$

$$V_1 := 2000$$
$$V_2 := 5000$$
$$V_3 := 100 \qquad \textbf{(2)}$$

$f(t) := 0.125;$

$$f := t \mapsto 0.125 \qquad \textbf{(3)}$$

$Pond1 := \frac{d}{dt} p_1(t) = \left(\frac{pf_3}{V_3}\right) \cdot p_3(t) - \left(\frac{pf_1}{V_1}\right) \cdot p_1(t) + f(t);$

$Pond2 := \frac{d}{dt} p_2(t) = \left(\frac{pf_1}{V_1}\right) \cdot p_1(t) - \left(\frac{pf_2}{V_2}\right) \cdot p_2(t);$

$Pond3 := \frac{d}{dt} p_3(t) = \left(\frac{pf_2}{V_2}\right) \cdot p_2(t) - \left(\frac{pf_3}{V_3}\right) \cdot p_3(t);$

$$Pond1 := \frac{d}{dt} p_1(t) = \frac{p_3(t)}{10} - \frac{p_1(t)}{100} + 0.125$$

$$Pond2 := \frac{d}{dt}\, p_2(t) = \frac{p_1(t)}{100} - \frac{p_2(t)}{500}$$

$$Pond3 := \frac{d}{dt}\, p_3(t) = \frac{p_2(t)}{500} - \frac{p_3(t)}{10} \quad \textbf{(4)}$$

Remember, $f(t) = 0.125$ lb/min for the first 48 hours or 2880 minutes.

$PondSystem := \{Pond1, Pond2, Pond3, p_1(0) = 0,\ p_2(0) = 0,\ p_3(0) = 0\};$

$$PondSystem := \left\{\frac{d}{dt}\, p_1(t) = \frac{p_3(t)}{10} - \frac{p_1(t)}{100} + 0.125, \frac{d}{dt}\, p_2(t) = \frac{p_1(t)}{100} - \frac{p_2(t)}{500}, \frac{d}{dt}\, p_3(t) = \frac{p_2(t)}{500} - \frac{p_3(t)}{10}, p_1(0) = 0, p_2(0) = 0, p_3(0) = 0\right\} \quad \textbf{(5)}$$

$PondSolution := dsolve(PondSystem);$

$$PondSolution := \left\{p_1(t) = \frac{500\left(\frac{51}{976} + \frac{1123\sqrt{479}}{467504}\right) e^{\frac{(-28+\sqrt{479})\,t}{500}}}{-28+\sqrt{479}} - \frac{500\left(\frac{51}{976} - \frac{1123\sqrt{479}}{467504}\right) e^{-\frac{(28+\sqrt{479})\,t}{500}}}{28+\sqrt{479}} + \frac{5\,t}{244} + \frac{63775}{7442}, p_2(t) = \frac{15625}{(-28+\sqrt{479})(28+\sqrt{479})} + \frac{1}{4(-28+\sqrt{479})(28+\sqrt{479})}\left(25\left(88\, e^{-\frac{(28+\sqrt{479})\,t}{500}}\sqrt{479}\left(\frac{51}{976} - \frac{1123\sqrt{479}}{467504}\right) - 88\, e^{\frac{(-28+\sqrt{479})\,t}{500}}\sqrt{479}\left(\frac{51}{976} + \frac{1123\sqrt{479}}{467504}\right) - 1976\left(\frac{51}{976} - \frac{1123\sqrt{479}}{467504}\right) e^{-\frac{(28+\sqrt{479})\,t}{500}} - 1976\left(\frac{51}{976} + \frac{1123\sqrt{479}}{467504}\right) e^{\frac{(-28+\sqrt{479})\,t}{500}} - \frac{127550}{61}\right.\right.\right. \quad \textbf{(6)}$$

$$- 5\, t\Bigg)\Bigg), p_3(t) = \frac{1275}{4\left(-28+\sqrt{479}\right)\left(28+\sqrt{479}\right)}$$

$$- \frac{1}{8\left(-28+\sqrt{479}\right)\left(28+\sqrt{479}\right)}\Bigg(400\, \mathrm{e}^{-\frac{\left(28+\sqrt{479}\right)t}{500}}\sqrt{479}\left(\frac{51}{976}-\frac{1123\sqrt{479}}{467504}\right)$$

$$- 400\, \mathrm{e}^{\frac{\left(-28+\sqrt{479}\right)t}{500}}\sqrt{479}\left(\frac{51}{976}+\frac{1123\sqrt{479}}{467504}\right) + 13200\Bigg(\frac{51}{976}$$

$$- \frac{1123\sqrt{479}}{467504}\Bigg)\mathrm{e}^{-\frac{\left(28+\sqrt{479}\right)t}{500}} + 13200\left(\frac{51}{976}+\frac{1123\sqrt{479}}{467504}\right)\mathrm{e}^{\frac{\left(-28+\sqrt{479}\right)t}{500}} + \frac{127550}{61}$$

$$+ 5\, t\Bigg)\Bigg\}$$

$P1 := eval\left(p_1(t), PondSolution\right);$

$$P1 := \frac{500\left(\frac{51}{976}+\frac{1123\sqrt{479}}{467504}\right)\mathrm{e}^{\frac{\left(-28+\sqrt{479}\right)t}{500}}}{-28+\sqrt{479}} \tag{7}$$

$$- \frac{500\left(\frac{51}{976}-\frac{1123\sqrt{479}}{467504}\right)\mathrm{e}^{-\frac{\left(28+\sqrt{479}\right)t}{500}}}{28+\sqrt{479}} + \frac{5\, t}{244} + \frac{63775}{7442}$$

$P2 := eval\left(p_2(t), PondSolution\right);$

$$P2 := \frac{15625}{\left(-28+\sqrt{479}\right)\left(28+\sqrt{479}\right)} \tag{8}$$

$$+ \frac{1}{4\left(-28+\sqrt{479}\right)\left(28+\sqrt{479}\right)}\Bigg(25\Bigg(88\, \mathrm{e}^{-\frac{\left(28+\sqrt{479}\right)t}{500}}\sqrt{479}\Bigg(\frac{51}{976}$$

$$- \frac{1123\sqrt{479}}{467504}\Bigg) - 88\, \mathrm{e}^{\frac{\left(-28+\sqrt{479}\right)t}{500}}\sqrt{479}\left(\frac{51}{976}+\frac{1123\sqrt{479}}{467504}\right) - 1976\Bigg(\frac{51}{976}$$

$$- \frac{1123\sqrt{479}}{467504}\Bigg)\mathrm{e}^{-\frac{\left(28+\sqrt{479}\right)t}{500}} - 1976\left(\frac{51}{976}+\frac{1123\sqrt{479}}{467504}\right)\mathrm{e}^{\frac{\left(-28+\sqrt{479}\right)t}{500}} - \frac{127550}{61}$$

$$- 5\, t \Big) \Big)$$

$P3 := eval\left(p_3(t), PondSolution\right);$

$$P3 := \frac{1275}{4\left(-28+\sqrt{479}\right)\left(28+\sqrt{479}\right)} - \frac{1}{8\left(-28+\sqrt{479}\right)\left(28+\sqrt{479}\right)}\Bigg(400\, \mathrm{e}^{-\frac{\left(28+\sqrt{479}\right)t}{500}} \sqrt{479}\left(\frac{51}{976} - \frac{1123\sqrt{479}}{467504}\right) - 400\, \mathrm{e}^{\frac{\left(-28+\sqrt{479}\right)t}{500}} \sqrt{479}\left(\frac{51}{976} + \frac{1123\sqrt{479}}{467504}\right) + 13200\left(\frac{51}{976} - \frac{1123\sqrt{479}}{467504}\right)\mathrm{e}^{-\frac{\left(28+\sqrt{479}\right)t}{500}} + 13200\left(\frac{51}{976} + \frac{1123\sqrt{479}}{467504}\right)\mathrm{e}^{\frac{\left(-28+\sqrt{479}\right)t}{500}} + \frac{127550}{61} + 5\, t\Bigg) \tag{9}$$

Change these values based on your assumptions for the initial system.

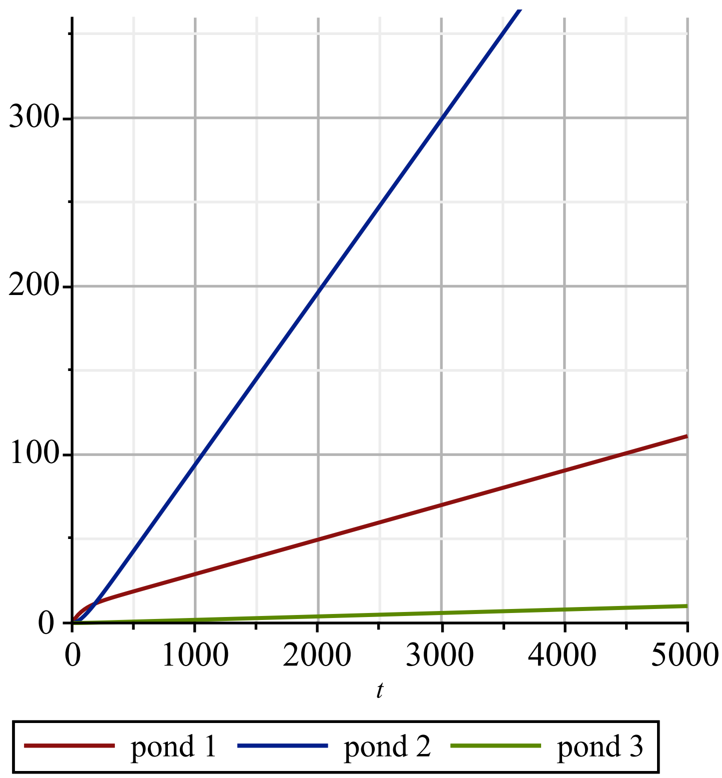
pollution_time := 5000 : *total_pollution* := 360 :

with(*DETools*) :

sol1 := *dsolve*(*eval*(*PondSystem*), *numeric*);

$$sol1 := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \tag{10}$$

plots:-*odeplot*(*sol1*, [[$t, p_1(t)$], [$t, p_2(t)$], [$t, p_3(t)$]], $t = 0$ ..*pollution_time*, *legend* = ["pond 1", "pond 2", "pond 3"], *gridlines*, *view* = [0 ..*pollution_time*, 0 ..*total_pollution*], *size* = [300, 300]);



evalf(eval(P1, t = 2880)); evalf(eval(P2, t = 2880)); evalf(eval(P3, t = 2880));

$$67.58599839 \tag{11}$$

$$286.7004839 \tag{11}$$

$$5.713517873 \tag{11}$$

pond1_48 := *evalf*(eval(P1, t = 2880));

$$pond1\_48 := 67.58599839 \tag{12}$$

pond2_48 := *evalf*(eval(P2, t = 2880));

$$pond2\_48 := 286.7004839 \tag{13}$$

pond3_48 := *evalf*(eval(P3, t = 2880));

$$pond3\_48 := 5.713517873 \tag{14}$$

The above three values will serve as the initial conditions for the long term system. First you need to remove the pollution input from pond 1.

Pond1 := *lhs*(*Pond1*) = *rhs*(*Pond1*) − 0.125;

$$Pond1 := \frac{d}{dt}\, p_1(t) = \frac{p_3(t)}{10} - \frac{p_1(t)}{100} \tag{15}$$

$PondSystemLT := \{Pond1, Pond2, Pond3, p_1(0) = \mathrm{pond1\_48},\ p_2(0) = pond2\_48,\ p_3(0) = pond3\_48\};$

$$PondSystemLT := \left\{ \frac{d}{dt}\, p_1(t) = \frac{p_3(t)}{10} - \frac{p_1(t)}{100},\ \frac{d}{dt}\, p_2(t) = \frac{p_1(t)}{100} - \frac{p_2(t)}{500},\ \frac{d}{dt}\, p_3(t) = \frac{p_2(t)}{500} - \frac{p_3(t)}{10}, p_1(0) = 67.58599839, p_2(0) = 286.7004839, p_3(0) = 5.713517873 \right\} \tag{16}$$

$PondSolutionLT := dsolve(PondSystemLT);$

$$PondSolutionLT := \left\{ p_1(t) = \frac{360000000163}{6100000000} + \left(\frac{816790469}{190625000} + \frac{572469260409\sqrt{479}}{2921900000000} \right) e^{\frac{(-28+\sqrt{479})\,t}{500}} + \left(-\frac{572469260409\sqrt{479}}{2921900000000} + \frac{816790469}{190625000} \right) e^{-\frac{(28+\sqrt{479})\,t}{500}}, p_2(t) = -\frac{27\left(-\frac{572469260409\sqrt{479}}{2921900000000} + \frac{816790469}{190625000} \right) e^{-\frac{(28+\sqrt{479})\,t}{500}}}{50} - \frac{27\left(\frac{816790469}{190625000} + \frac{572469260409\sqrt{479}}{2921900000000} \right) e^{\frac{(-28+\sqrt{479})\,t}{500}}}{50} + \frac{\left(-\frac{572469260409\sqrt{479}}{2921900000000} + \frac{816790469}{190625000} \right) e^{-\frac{(28+\sqrt{479})\,t}{500}} \sqrt{479}}{50} - \frac{e^{\frac{(-28+\sqrt{479})\,t}{500}} \sqrt{479} \left(\frac{816790469}{190625000} + \frac{572469260409\sqrt{479}}{2921900000000} \right)}{50} + \frac{360000000163}{1220000000}, \right. \tag{17}$$

$$p_3(t) = -\frac{\left(-\frac{572469260409\sqrt{479}}{2921900000000} + \frac{816790469}{190625000}\right) e^{-\frac{\left(28+\sqrt{479}\right)t}{500}} \sqrt{479}}{50}$$

$$+ \frac{e^{\frac{\left(-28+\sqrt{479}\right)t}{500}} \sqrt{479} \left(\frac{816790469}{190625000} + \frac{572469260409\sqrt{479}}{2921900000000}\right)}{50}$$

$$- \frac{23\left(-\frac{572469260409\sqrt{479}}{2921900000000} + \frac{816790469}{190625000}\right) e^{-\frac{\left(28+\sqrt{479}\right)t}{500}}}{50}$$

$$- \frac{23\left(\frac{816790469}{190625000} + \frac{572469260409\sqrt{479}}{2921900000000}\right) e^{\frac{\left(-28+\sqrt{479}\right)t}{500}}}{50} + \frac{360000000163}{61000000000}\Bigg\}$$

$P1LT := eval\left(p_1(t), PondSolutionLT\right);$

$$P1LT := \frac{360000000163}{6100000000} + \left(\frac{816790469}{190625000} + \frac{572469260409\sqrt{479}}{2921900000000}\right) e^{\frac{\left(-28+\sqrt{479}\right)t}{500}} + \left(-\frac{572469260409\sqrt{479}}{2921900000000} + \frac{816790469}{190625000}\right) e^{-\frac{\left(28+\sqrt{479}\right)t}{500}} \quad \textbf{(18)}$$

$P2LT := eval\left(p_2(t), PondSolutionLT\right);$

$$P2LT := -\frac{27\left(-\frac{572469260409\sqrt{479}}{2921900000000} + \frac{816790469}{190625000}\right) e^{-\frac{\left(28+\sqrt{479}\right)t}{500}}}{50} \quad \textbf{(19)}$$

$$- \frac{27\left(\frac{816790469}{190625000} + \frac{572469260409\sqrt{479}}{2921900000000}\right) e^{\frac{\left(-28+\sqrt{479}\right)t}{500}}}{50}$$

$$+ \frac{\left(-\frac{572469260409\sqrt{479}}{2921900000000} + \frac{816790469}{190625000}\right) e^{-\frac{\left(28+\sqrt{479}\right)t}{500}} \sqrt{479}}{50}$$

$$-\frac{e^{\frac{(-28+\sqrt{479})t}{500}}\sqrt{479}\left(\frac{816790469}{190625000}+\frac{572469260409\sqrt{479}}{2921900000000}\right)}{50}+\frac{360000000163}{1220000000}$$

$P3LT := eval\left(p_3(t), PondSolutionLT\right);$

$$P3LT := -\frac{\left(-\frac{572469260409\sqrt{479}}{2921900000000}+\frac{816790469}{190625000}\right)e^{-\frac{(28+\sqrt{479})t}{500}}\sqrt{479}}{50}$$
$$+\frac{e^{\frac{(-28+\sqrt{479})t}{500}}\sqrt{479}\left(\frac{816790469}{190625000}+\frac{572469260409\sqrt{479}}{2921900000000}\right)}{50}$$
$$-\frac{23\left(-\frac{572469260409\sqrt{479}}{2921900000000}+\frac{816790469}{190625000}\right)e^{-\frac{(28+\sqrt{479})t}{500}}}{50}$$
$$-\frac{23\left(\frac{816790469}{190625000}+\frac{572469260409\sqrt{479}}{2921900000000}\right)e^{\frac{(-28+\sqrt{479})t}{500}}}{50}+\frac{360000000163}{61000000000} \quad \textbf{(20)}$$

evalf(eval(P1LT, t = 2000)); evalf(eval(P2LT, t = 2000)); evalf(eval(P3LT, t = 2000));

59.01639347

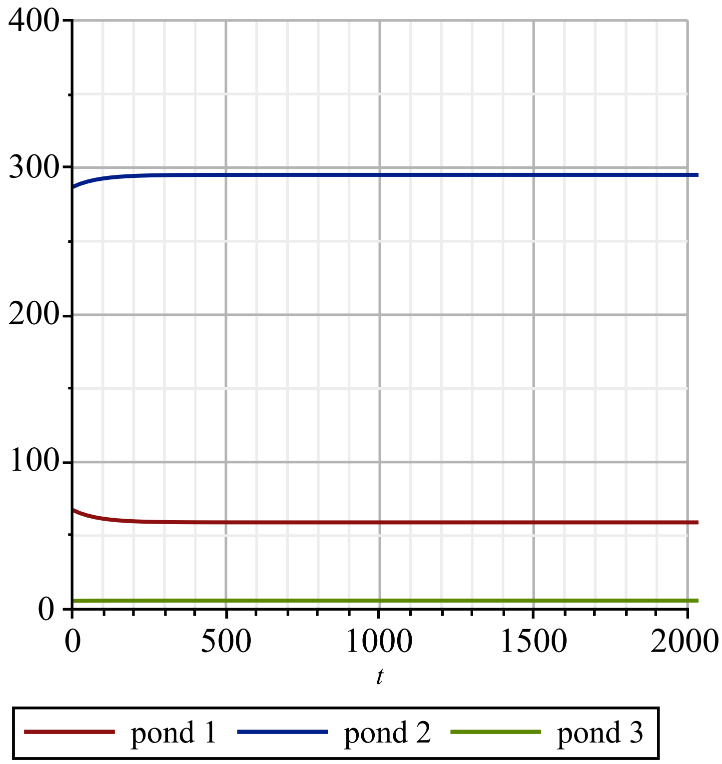
295.0819673

5.901639347 **(21)**

$sol2 := dsolve(eval(PondSystemLT), numeric);$

$sol2 :=$ **proc**(x_rkf45) ... **end proc** **(22)**

$plots\text{:-}odeplot\left(sol2, \left[[t, p_1(t)], [t, p_2(t)], [t, p_3(t)] \right], t = 0..5000, legend = ["pond 1", "pond 2", "pond 3"], gridlines, view = [0..2000, 0..400], size = [300, 300]\right);$



So what is happening LT in this system? Well let's look at each pond's LT amount of pollution, its size, and the flow out of it.

$$\frac{\text{evalf}(\text{eval}(\text{P1LT}, \text{t} = 2000))}{V_1} \cdot pf_1;$$

$$0.5901639348 \tag{23}$$

$$\frac{\text{evalf}(\text{eval}(\text{P2LT}, \text{t} = 2000))}{V_2} \cdot pf_2;$$

$$0.5901639346 \tag{24}$$

$$\frac{\text{evalf}(\text{eval}(\text{P3LT}, \text{t} = 2000))}{V_3} \cdot pf_3;$$

$$0.5901639347 \tag{25}$$

The units here come out to be $\frac{lb}{\text{min}}$ so the LT system equalizes the net flow of pollution.