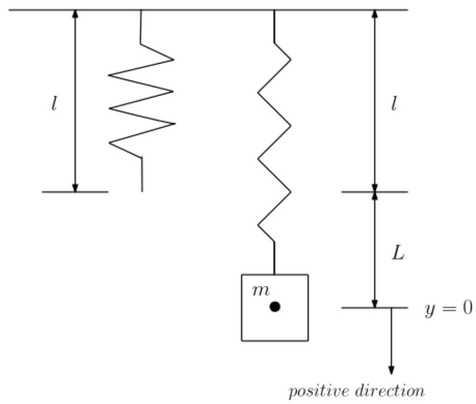
THE MASS SPRING PROBLEM

Consider an unstreched spring of length l.



Now, attach an object of mass m to the spring. It will be streteched downward a length L. This stretched length will be our starting point for analysis. Let this point be defined as $y = 0$ with a positive direction in the downward direction.

From Newton's second law we have $F = m \cdot a$ or in our case $F = m \cdot y''$. What are the forces that can act on this object?

1. Gravity
The force of gravity will always act on this object and will be defined as

$$F_g = m \cdot g.$$

2. Spring
Hooke's law will govern the force on the spring.

$$F_s = -k \cdot (L + y)$$

3. Damping
Damping will resist any movement and will be proportional to the velocity of the object.

$$F_d = -\gamma \cdot y'$$

4. External forces
In this term you can put in any other forces acting on the object as a function of time.

$$F(t)$$

Now we add all of these forces into our Newton's Second Law equation.

$$m \cdot y'' = m \cdot g - k \cdot (L + y) - \gamma \cdot y' + F(t)$$

$$m \cdot y'' + \gamma \cdot y' + k \cdot y = m \cdot g - k \cdot L + F(t)$$

But, at equilibrium the system is at rest and $m \cdot g = k \cdot L$. This can be used to determine k. So we can rewrite our equation as

$$m \cdot y'' + \gamma \cdot y' + k \cdot y = F(t) \; .$$

and it will have the following initial conditions.

$y(0) = y0$ Initial displacement from the at rest, equilibrium position.
$y'(0) = (y0)'$ Initial velocity.

$restart;$
$with(DETools):$
$eq1 := diff(y(t), t) = v(t);$

$$eq1 := \frac{d}{dt}\, y(t) = v(t) \qquad \textbf{(1)}$$

$eq2 := diff(v(t), t) = -\left(\frac{k}{m}\right) \cdot y(t) - \left(\frac{b}{m}\right) \cdot v(t);$

$$eq2 := \frac{d}{dt}\, v(t) = -\frac{k\, y(t)}{m} - \frac{b\, v(t)}{m} \qquad \textbf{(2)}$$

$sys := [eq1, eq2];$

$$sys := \left[\frac{d}{dt}\, y(t) = v(t), \frac{d}{dt}\, v(t) = -\frac{k\, y(t)}{m} - \frac{b\, v(t)}{m}\right] \qquad \textbf{(3)}$$

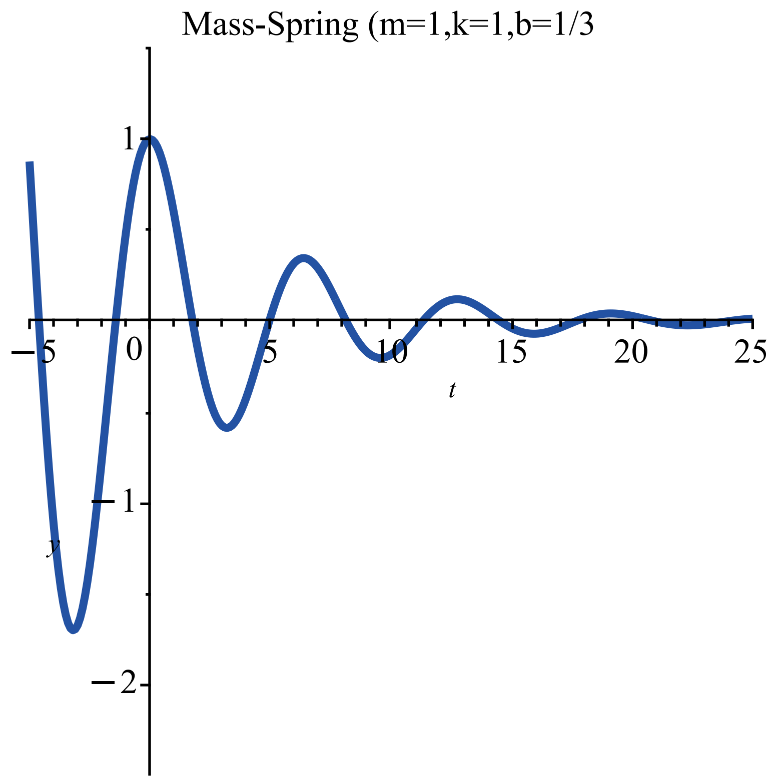
$sys1 := subs\left(b = \frac{1}{3}, m = 1, k = 1, sys\right);$

$$sys1 := \left[\frac{d}{dt}\, y(t) = v(t), \frac{d}{dt}\, v(t) = -y(t) - \frac{v(t)}{3}\right] \qquad \textbf{(4)}$$

$ics := [[0, 1, 0]];$

$$ics := [[0, 1, 0]] \qquad \textbf{(5)}$$

$DEplot\left(sys1,\ [y(t), v(t)],\ t=-5..25,\ ics,\ y=-2.5..1.5,\ title=\text{"Mass-Spring (m=1,k=1,b=1/3"},\ stepsize=0.1,\ scene=[t,y],\ size=[300,300]\right);$

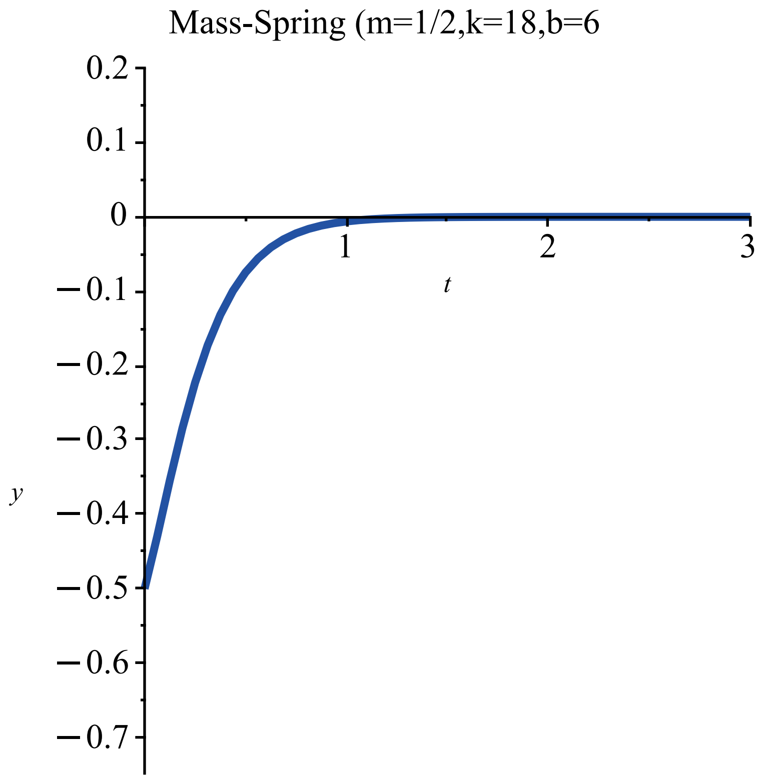


Now, let's put in different initial conditions to create a critically damped system.

$sys2 := subs\left(b=6,\ m=\frac{1}{2},\ k=18,\ sys\right);\ ics2 := \left[\left[0,-\frac{1}{2},1\right]\right]:$

$$sys2 := \left[\frac{d}{dt}\,y(t)=v(t),\ \frac{d}{dt}\,v(t)=-36\,y(t)-12\,v(t)\right] \quad \textbf{(6)}$$

$DEplot(sys2, [y(t), v(t)], t = 0 .. 3, ics2, y = -0.75 .. 0.2, title = $"Mass-Spring (m=1/2,k=18,b=6"$,$
$stepsize = 0.1, scene = [t, y], size = [300, 300]);$



With Maple it is easy to see the solution but what does this equation look like.

First, explicitly define the initial conditions.

$inits := y(0) = -0.5, v(0) = 1;$

$$inits := y(0) = -0.5, v(0) = 1 \quad \textbf{(7)}$$

Then, define the equations.

$eqs := \frac{d}{dt} y(t) = v(t), \frac{d}{dt} v(t) = -36\, y(t) - 12\, v(t);$

$$eqs := \frac{d}{dt} y(t) = v(t), \frac{d}{dt} v(t) = -36\, y(t) - 12\, v(t) \quad \textbf{(8)}$$

Now solve the system.

$dsolve(\{eqs, inits\}, \{y(t), v(t)\});$

$$\left\{v(t) = -e^{-6t}(-12t - 1), y(t) = e^{-6t}\left(-2t - \frac{1}{2}\right)\right\} \quad (9)$$

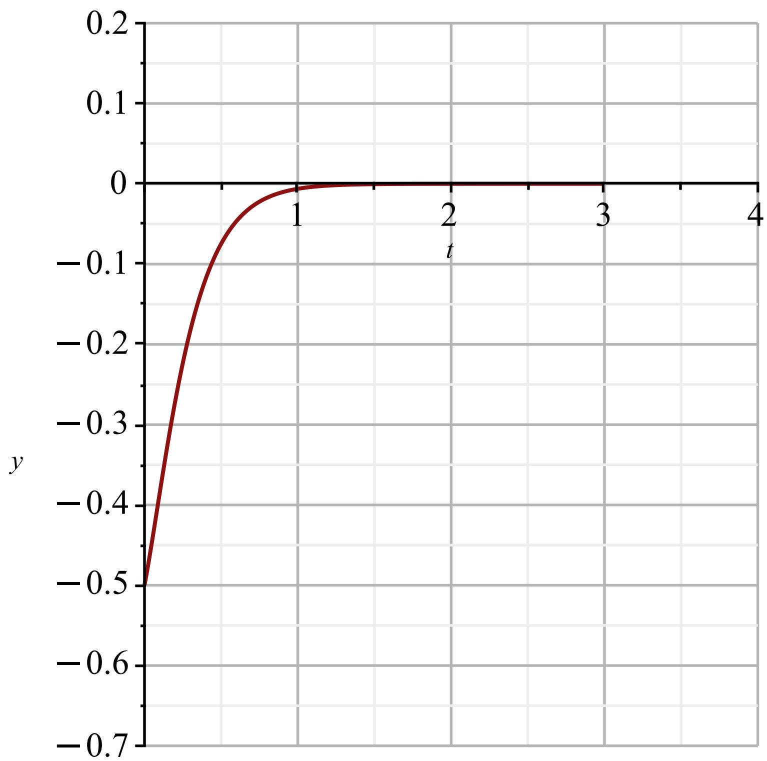
Here you can see what the position $y(t)$ looks like.

Now let's plot it. First get a numeric solution to the system so you can plot $y(t)$.

$sol\_numeric := dsolve(eval(\{eqs, inits\}), numeric);$

$$sol\_numeric := \textbf{proc}(x\_rkf45) \ \ldots \ \textbf{end proc} \quad (10)$$

$plots\text{:-}odeplot(sol\_numeric, [t, y(t)], t = 0..3, view = [0..4, -0.7..0.2], gridlines, size = [300, 300]);$



Why is the above example referred to as critical damping. We need to look at our second order equation to find out.

$$m \cdot y'' + \gamma \cdot y' + k \cdot y = 0$$

Let's solve it.

$f := y \rightarrow m \cdot y^2 + \gamma \cdot y + k;$

$$f := y \mapsto m \cdot y^2 + \gamma \cdot y + k \tag{11}$$

$solve(f(y) = 0, y);$

$$\frac{-\gamma + \sqrt{\gamma^2 - 4\,m\,k}}{2\,m}, -\frac{\gamma + \sqrt{\gamma^2 - 4\,m\,k}}{2\,m} \tag{12}$$

We see the expected result for the solution of a quadratic equation. What is interesting to us is what is happening under the square root sign. Three cases make sense.

1. $\gamma^2 - 4\,mk$ is equal to zero.

For our examples above m is m, b is γ, and k is k. So for the example immediately above $m = 1/2,\ k = 18,\ b = 6$

$$m \tag{13}$$

$\sqrt{\gamma^2 - 4\,mk}$ is $\sqrt{6^2 - 4 \cdot \left(\frac{1}{2}\right) \cdot 18}$ or $\sqrt{36 - 36}$ or zero. So that is why the system was critically damped.

Note that since $\gamma^2 - 4\,mk = 0$ then $\gamma^2 = 4\,mk$ and $\gamma = 2\sqrt{mk}$ which will be referred to as the critical damping coefficient.

$$\gamma^2 = 4\,mk \tag{14}$$

2. $\gamma^2 - 4\,mk$ is greater than zero.

Here, γ will be greater than $2\sqrt{mk}$ or the critical damping we defined in 1.

3. $\gamma^2 - 4\,mk$ is less than zero.

Here, γ will be less than $2\sqrt{mk}$ or the critical damping we defined in 1.

The problem set up for the above criticalled damped system started with a 16 lb object that stretches a spring 8/9 feet.

Add a damper that will exert a force of 12 lbs when the velocity is 2 ft/sec. Remember our equation is

$$m \cdot y'' + \gamma \cdot y' + k \cdot y = F(t) \text{ with } F(t) = 0.$$

So we have $12 = \gamma \cdot 2 \Rightarrow \gamma = 6$. (This was b in the example).

The mass is $m = \frac{W}{g} = \frac{16}{32} = \frac{1}{2}$.

Remember $m \cdot g = k \cdot L$ so

$$k = \frac{(m \cdot g)}{L} = \frac{16}{\left(\frac{8}{9}\right)} = 18.$$

The spring is displace 6 inches upwards and given an initial velocity of 1 ft/sec. Since downward is positive the initial conditions are

$$y(0) = -\frac{1}{2} \text{ and } (y(0)' = 1.$$

Now let's create an overdamped system. Let the damper exert a force of 17 lbs this time. Now we have $17 = \gamma \cdot 2 \Rightarrow \gamma = 8.5$. The γ is greater than the critical damping factor which was 6. Now we have

$$m\left(\frac{d^2}{dx^2} y(x)\right) + \gamma\left(\frac{d}{dx} y(x)\right) + k\,y(x) = 0$$

$$\frac{1}{2}\left(\frac{d^2}{dx^2} y(x)\right) + 8.5\left(\frac{d}{dx} y(x)\right) + 18\,y(x) = 0$$

Now let's multiply through by 2 and get

$$\left(\frac{d^2}{dx^2} y(x)\right) + 17\left(\frac{d}{dx} y(x)\right) + 36\, y(x) = 0$$

$$\frac{d^2}{dx^2} y(x) + 17 \frac{d}{dx} y(x) + 36\, y(x) = 0 \quad \textbf{(15)}$$

$f2 := y \rightarrow y^2 + 17 \cdot y + 36;$

$$f2 := y \mapsto y^2 + 17 \cdot y + 36 \quad \textbf{(16)}$$

$evalf(solve(f2(y) = 0, y));$

$$-2.479202710, -14.52079729 \quad \textbf{(17)}$$

So there are 2 real roots.

Again, explicitly define the initial conditions.

$inits := y(0) = -0.5, v(0) = 1;$

$$inits := y(0) = -0.5, v(0) = 1 \quad \textbf{(18)}$$

Then, define the system equations.

$eqs2 := \frac{d}{dt} y(t) = v(t), \frac{d}{dt} v(t) = -17 \cdot v(t) - 36\, y(t);$

$$eqs2 := \frac{d}{dt} y(t) = v(t), \frac{d}{dt} v(t) = -17\, v(t) - 36\, y(t) \quad \textbf{(19)}$$

Now solve the system.

$evalf(dsolve(\{eqs2, inits\}, \{y(t), v(t)\}));$

$$\{v(t) = 1.288932058\, e^{-2.479202710\, t} - 0.2889320583\, e^{-14.52079729\, t}, y(t) = -0.5198978095\, e^{-2.479202710\, t} + 0.0198978095\, e^{-14.52079729\, t}\} \quad \textbf{(20)}$$

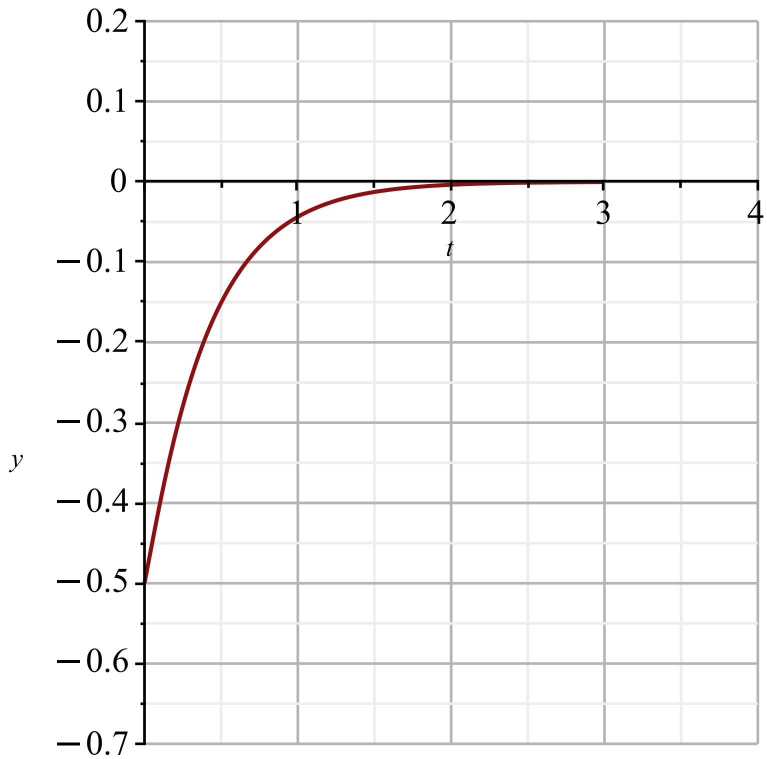
Here you can see what the position $y(t)$ looks like.

Now let's plot it. First get a numeric solution to the system so you can plot $y(t)$.

$sol\_numeric2 := dsolve(eval(\{eqs2, inits\}), numeric);$

$$sol\_numeric2 := \mathbf{proc}(x\_rkf45) \ \ldots \ \mathbf{end\ proc} \quad \mathbf{(21)}$$

$plots{:-}odeplot(sol\_numeric2, [t, y(t)], t = 0..3, view = [0..4, -0.7..0.2], gridlines, size = [300, 300]);$



In this case the overdamped case takes a little longer to settle to 0. Note that the crtically damped system settles as quickly as possible without overshooting the steady state condition, 0 here. What does an underdamped system look like?

To create an underdamped system γ must be less than the critical value, 6 in these examples. Let the damper exert 5 lbs when the velocity is 2 ft/sec.

Now we have $5 = \gamma \cdot 2 \Rightarrow \gamma = 2.5$.

The γ is less than the critical damping factor which was 6. Now we have

$$m\left(\frac{d^2}{dx^2}\, y(x)\right) + \gamma\left(\frac{d}{dx}\, y(x)\right) + k\, y(x) = 0$$

$$\frac{1}{2}\left(\frac{d^2}{dx^2}\,y(x)\right)+2.5\left(\frac{d}{dx}\,y(x)\right)+18\,y(x)=0$$

Now let's multiply through by 2 and get

$$\frac{d^2}{dx^2}\,y(x)\Bigg)+5\cdot\left(\frac{d}{dx}\,y(x)\right)+36\,y(x)=0$$

$f2 := y \rightarrow y^2 + 5\cdot y + 36$;

$$f2 := y \mapsto y^2 + 5\cdot y + 36 \qquad \textbf{(22)}$$

$evalf(solve(f2(y)=0, y))$;

$$-2.500000000 + 5.454356055\,\mathrm{I},\ -2.500000000 - 5.454356055\,\mathrm{I} \qquad \textbf{(23)}$$

So there are 2 complex roots so we expect some sort of sinusoidal response.

Again, explicitly define the initial conditions.

$inits := y(0) = -0.5, v(0) = 1$;

$$inits := y(0) = -0.5,\ v(0) = 1 \qquad \textbf{(24)}$$

Then, define the system equations.

$eqs3 := \frac{d}{dt}\,y(t) = v(t),\ \frac{d}{dt}\,v(t) = -5\cdot v(t) - 36\,y(t)$;

$$eqs3 := \frac{d}{dt}\,y(t) = v(t),\ \frac{d}{dt}\,v(t) = -5\,v(t) - 36\,y(t) \qquad \textbf{(25)}$$

Now solve the system.

$evalf(dsolve(\{eqs3, inits\}, \{y(t), v(t)\}))$;

$$\{v(t) = -0.5000000000\,e^{-2.500000000\,t}\,(-5.683530680\,\sin(5.454356055\,t) - 2.\,\cos(5.454356055\,t)),\ y(t) = e^{-2.500000000\,t}\,(-0.04583492483\,\sin(5.454356055\,t) - 0.5000000000\,\cos(5.454356055\,t))\} \qquad \textbf{(26)}$$

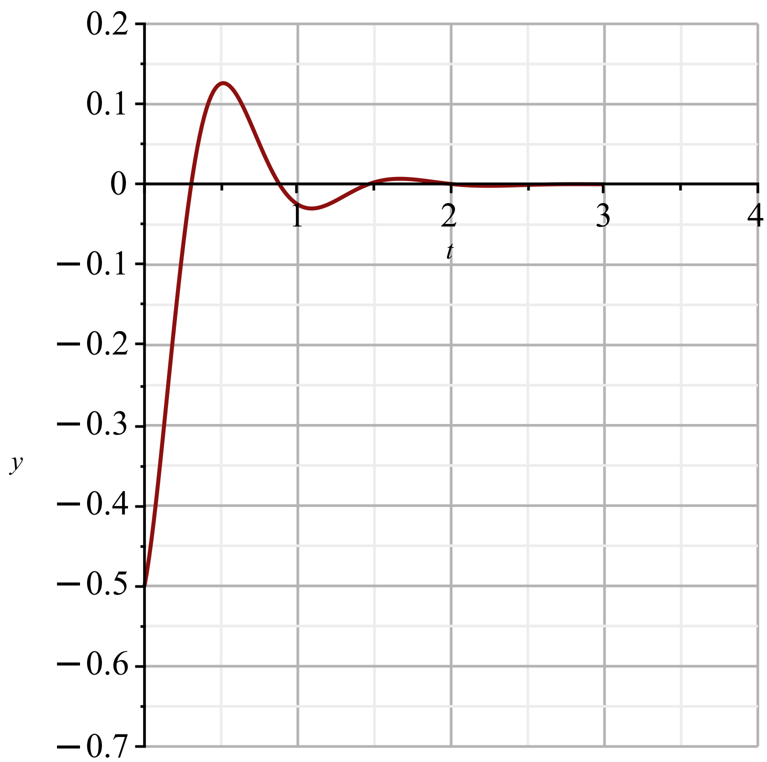
Here you can see what the position $y(t)$ looks like.

Now let's plot it. First get a numeric solution to the system so you can plot $y(t)$.

$sol\_numeric3 := dsolve(eval(\{eqs3, inits\}), numeric);$

$$sol\_numeric3 := \mathbf{proc}(x\_rkf45) \ \ldots \ \mathbf{end\ proc} \tag{27}$$

$plots\text{:-}odeplot(sol\_numeric3, [t, y(t)], t = 0 ..3, view = [0 ..4, -0.7 ..0.2], gridlines, size = [300, 300]);$



The overshoot is clearly seen in this latest plot of an underdamped system.