THE LOGISTIC EQUATION AND POPULATION GROWTH

The banking example in the introduction was a simple growth model. Leaving the funds in a bank you expect the growth to continue forwever. Let's come up with an example with lots of growth.

restart
$growth := diff(P(t), t) = 0.5 \cdot P(t);$

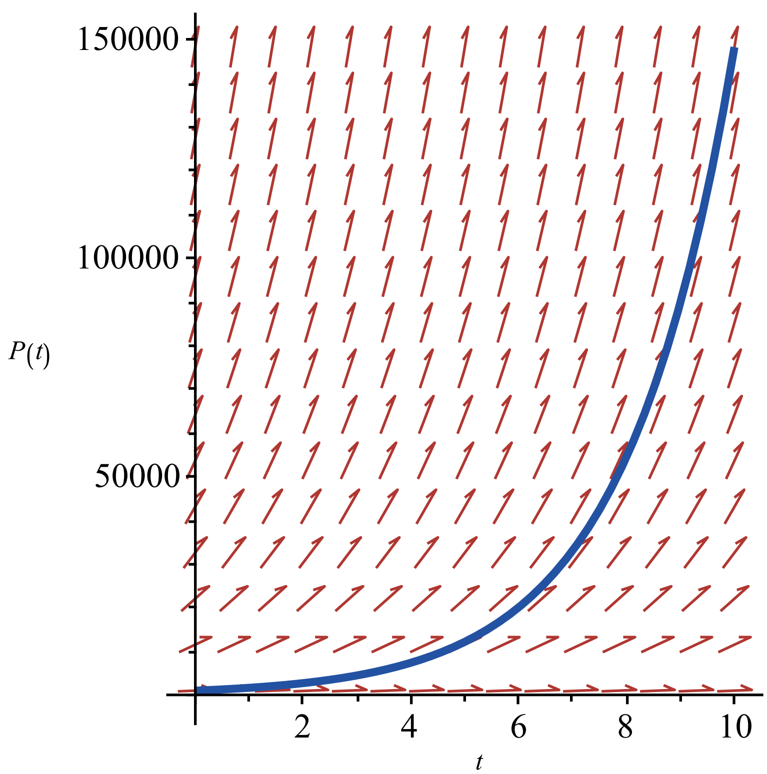
$$growth := \frac{d}{dt} P(t) = 0.5\, P(t) \quad \textbf{(1)}$$

$ans1 := dsolve(\{P(0) = 1000, \textbf{(1)}\});$

$$ans1 := P(t) = 1000\, e^{\frac{t}{2}} \quad \textbf{(2)}$$

$with(plots):$
$with(DETools):$

$ics := [P(0) = 1000]:$
$DEplot(growth, P(t), t = 0..10, ics, dirfield = [15, 15], size = [300, 300]);$



This simple growth model looks familiar but is not very realistic for population growth. Populations don't continue to grow without limit.

There are many sources of the logistic equation. This one is from https://sites.radford.edu/~npsigmon/courses/calculus3/mword/Section7.5notes.pdf. Nice and simple and this link includes harvesting and extinction.

<u>Logistic Population Growth Model</u>

The initial value problem for logistic population growth,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right), \quad P(0) = P_0,$$

has solution

$$P(t) = \frac{K}{1 + Ae^{-kt}} \text{ where } A = \frac{K - P_0}{P_0}.$$

Here,

t = the time the population grows
P or $P(t)$ = the population after time t.
k = relative growth rate coefficient
K = *carrying capacity*, the amount that when exceeded will result in the population decreasing.
P_0 = *initial population*, or the population we start with at time $t = 0$, that is, $P(0) = P_0$.

Let's start with the relative growth coeffieient $k = 0.3$ and the carrying capacity $K = 10000$. We will let Maple solve the differential equation and then plot it.

restart
with(*DEtools*) : *with*(*plots*) :

$de := diff(P(t), t) = 0.3 * P(t) * (1 - P(t)/10000);$

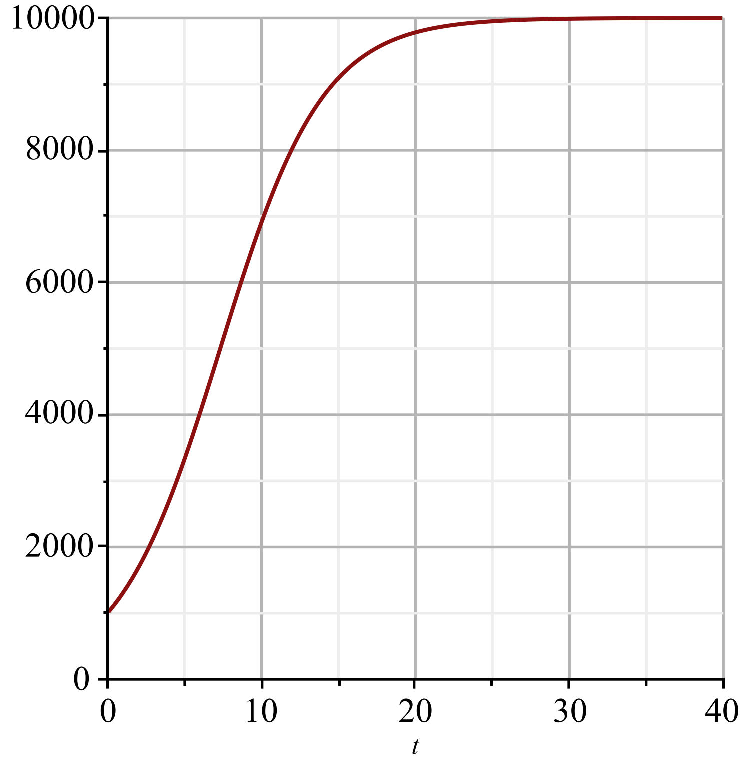
$$de := \frac{d}{dt} P(t) = 0.3\, P(t) \left(1 - \frac{P(t)}{10000}\right) \quad \textbf{(3)}$$

To solve it, we will start with an initial population of 1000.

$ans1 := dsolve(\{P(0) = 1000, de\})$

$$ans1 := P(t) = \frac{10000}{1 + 9\,\mathrm{e}^{-\frac{3t}{10}}} \qquad \textbf{(4)}$$

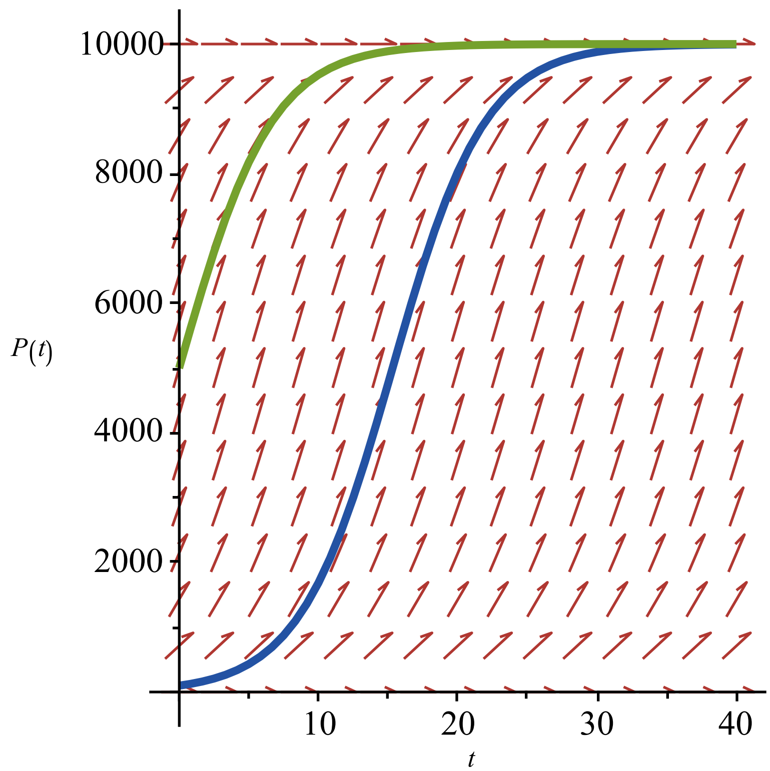
$plot(rhs(ans1), t = 0..40, size = [300, 300], gridlines, view = [0..40, 0..10000]);$



This is what a logistic growth equation looks like given a specific carrying capacity.

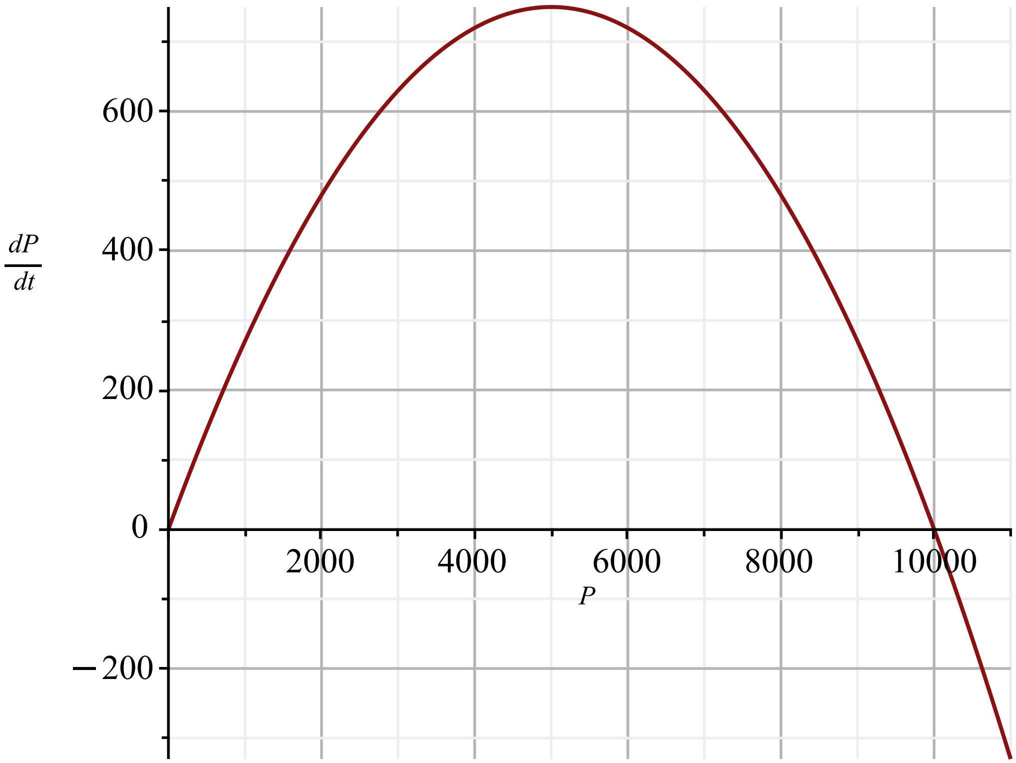
Now let's look at the direction field for our equation and try 100 and 5000 as the initial populations.

$DEplot(\text{de}, P(t), t = 0..40, P = 0..10000, \{[P(0) = 100], [P(0) = 5000]\}, size = [300, 300],$
$dirfield = [15, 15])$



Remember that the right hand side of equation (3) calculates the slope of $P(t)$ versus time t. What if we simply plot this $\frac{dP}{dt}$ versus P?

$plot(rhs(de), P = 0 ..11000, gridlines, size = [400, 300]);$



This interesting plot shows two points where $\frac{dP}{dt}$ is zero. The first point is at zero and the second is at 10,000 or the carrying capacity that we set above. The rate of change is greatest at 5,000 or the carrying capacity divided by 2.

To find when the population reaches 8000, we just set the right hand side of equation (4) equal to 8000 and evaluate the expression at that point.

$Pop8000 := evalf(solve(rhs(ans1) = 8000, t))$

$$Pop8000 := 11.94506313 \quad \textbf{(5)}$$

Now suppose that we are really modeling the population of fish in a small lake. Further we want to harvest a certain number of them every year. How should change our model?

Think of the harvesting as a rate per year. Then the harvesting would be expressed as a fixed rate per year. Now, rewrite equation (3) as

$deH := diff(P(t), t) = 0.3 * P(t) * (1 - P(t)/10000) - 400;$ =

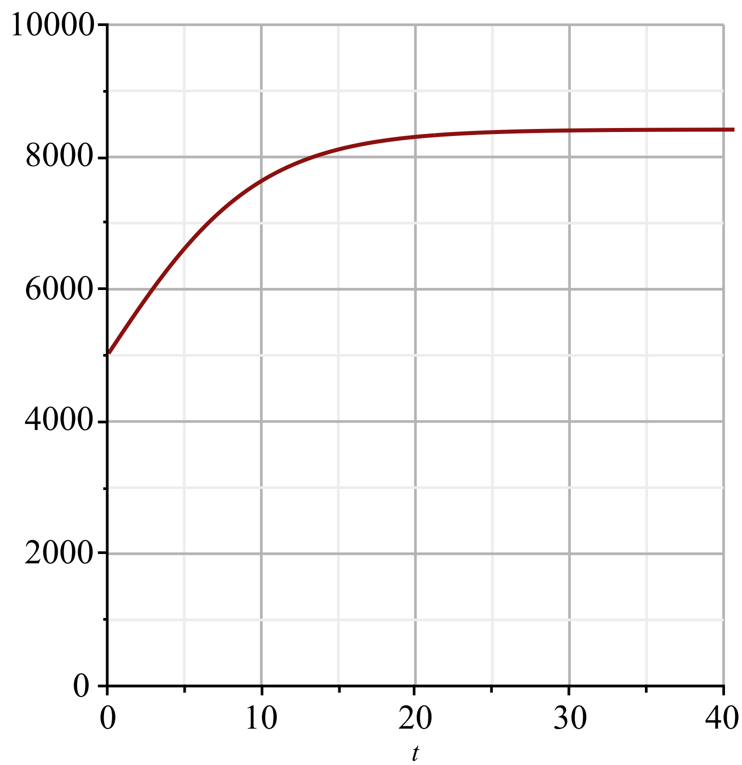
$$\frac{d}{dt} P(t) = 0.3 P(t) \left(1 - \frac{1}{10000} P(t)\right) - 400$$

To solve it, we will start with an initial population of 1000.

$ans2 := dsolve(\{P(0) = 5000, deH\})$

$$ans2 := P(t) = 5000 + \frac{1000 \sqrt{105} \tanh\left(\frac{\sqrt{105}\, t}{100}\right)}{3} \tag{6}$$

$plot(rhs(ans2), t = 0..50, size = [300, 300], gridlines, view = [0..40, 0..10000]);$



So to find the new steady state population of fish in the lake, plug 40 into equation (6).

$evalf(subs(t = 40, rhs(ans2)))$

$$8413.769950 \tag{7}$$

Now, what happens is there is a minimum population that maintains viability. If the population is below, say 1000, then the population will become extinct

$$deMin := diff(P(t), t) = 0.3 * P(t) * (1 - P(t) / 10000) \cdot \left(1 - \frac{1000}{P(t)}\right); =$$

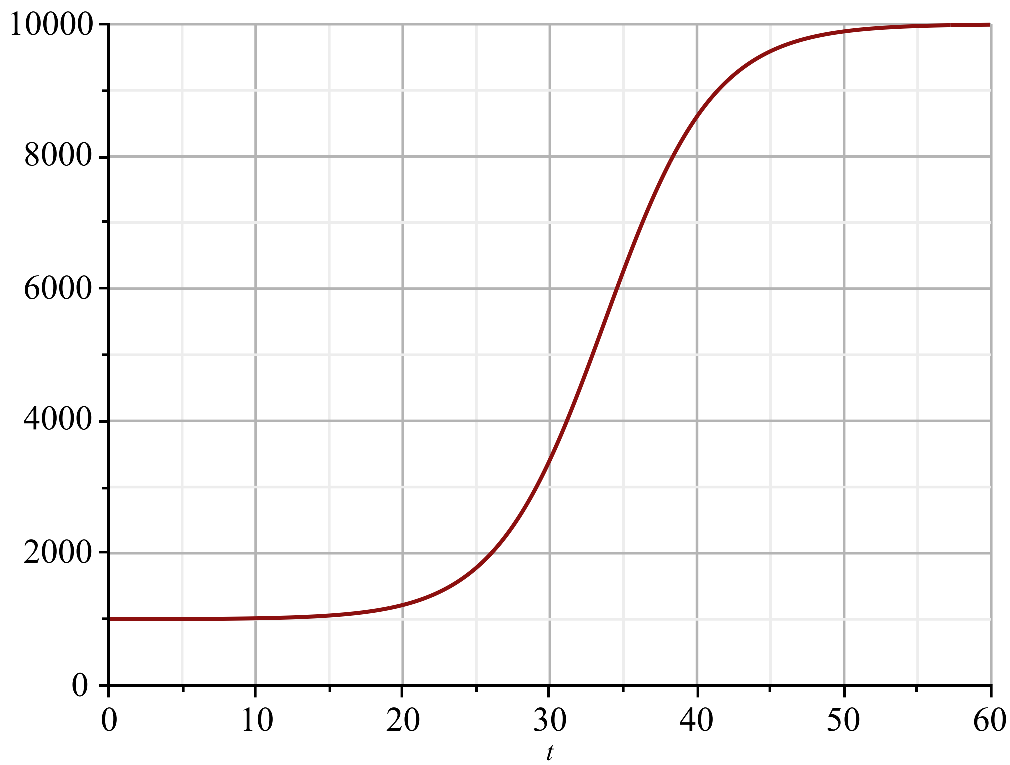
$$\frac{d}{dt} P(t) = 0.3\, P(t) \left(1 - \frac{1}{10000} P(t)\right) \left(1 - \frac{1000}{P(t)}\right)$$

To solve it, we will start with an initial population of 1001.

$$ans2 := dsolve(\{P(0) = 1001, deMin\});$$

$$ans2 := P(t) = \frac{10000\, e^{\frac{27 t}{100}} + 8999000}{8999 + e^{\frac{27 t}{100}}} \quad \textbf{(8)}$$

$$plot(rhs(ans2), t = 0..60, size = [400, 300], gridlines, view = [0..60, 0..10000]);$$



Even when starting with just over the minimum viable population, the population will still stabilize at its carrying capacity.

Now, let's plot several initial populations - 999, 1000, 1001, and 5000. Again, the solutions clearly follow the direction field curves form each initial condition.

$DEplot(deMin, P(t), t = 0..60, P = 0..10000, \{[P(0) = 999], [P(0) = 5000], [P(0) = 1000], [P(0) = 1001]\}, size = [400, 300], dirfield = [15, 15])$